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RATIONAL CHEBYSHEV APPROXIMATION TO CERTAIN ENTIRE FUNCTIONS OF ZERO ORDER ON THE POSITIVE REAL AXIS. II

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ABSTRACT. A sufficient condition is given on the growth of certain entire functions which guarantee a certain rate of convergence of the error in approximating reciprocals of a class of entire functions by reciprocals of polynomials under the uniform norm on the positive real axis.

Recently we proved

THEOREM 1 [1, THEOREM 7]. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$ *and* $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of zero order satisfying the assumptions that*

$$1 < \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 = \rho_l < \infty \tag{1}$$

and

$$0 < \liminf_{r\to\infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = b_l, \qquad \limsup_{r\to\infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = B_l < \infty, \tag{2}$$

where $M(r) = \max_{|z|=r}|f(z)|$. *Then there exists a sequence of polynomials* $\{P_n(x)\}_{n=0}^{\infty}$ *for which*

$$\lim_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} = 0. \tag{3}$$

One sees that (2) implies the limit in (1), for suppose, to the contrary,

$$0 < \liminf_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \sigma < \Lambda + 1$$

$$= \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} < \infty.$$

Then for each $\epsilon > 0$, there is a sequence of values of r for which

$$\log M(r) \leqslant (\log r)^{\sigma+\epsilon}. \tag{4}$$

From (4) we obtain

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$$(\log M(r))/(\log r)^{\Lambda+1} \leqslant (\log r)^{\sigma+\epsilon}/(\log r)^{\Lambda+1},$$

and we may choose ϵ so that $\sigma + \epsilon < \Lambda + 1$. Then clearly we have

$$\liminf_{r\to\infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = b_l = 0,$$

which contradicts (2).

On the other hand the limit in (1) may not imply (2). For example let

$$f_1(z) = 1 + \sum_{n=2}^{\infty} \frac{z^n}{\exp[(n \log n)^2]},$$

$$f_2(z) = 1 + \sum_{n=2}^{\infty} \frac{z^n}{\exp[(n/\log n)^2]}.$$

It is easy to calculate for

$$f_1(z), \quad \Lambda = 1 = \sigma - 1, \quad B_l = 0, \qquad f_2(z), \quad \Lambda = 1 = \sigma - 1, \quad B_l = \infty.$$

Hence it is natural to ask whether one can get (3) under the only assumption

$$0 < \lim_{r\to\infty} \frac{\log \log M(r)}{\log \log r} = \Lambda + 1 < \infty. \tag{5}$$

In fact we prove the more general

THEOREM 2. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of zero order satisfying the assumption that*

$$0 < \Lambda < \liminf_{r\to\infty} \frac{\log \log M(r)}{\log \log r} = \sigma < \infty.$$

Then there exists a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ for which

$$\lim_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} = 0.$$

REMARK. The method used here is much simpler than the one used in [1].

PROOF. By assumption

$$\limsup_{r\to\infty} \frac{\log \log M(r)}{\log \log r} = \Lambda + 1.$$

Then it is known [2, Theorems 1 and 3] that

$$\limsup_{n\to\infty} \frac{\log n}{\log\{n^{-1} \log |a_n|^{-1}\}} = \Lambda. \tag{6}$$

From (6) it is easy to get for each $\epsilon > 0$ there is an n_0 depending on ϵ such that for all $n \geqslant n_0(\epsilon)$,

$$|a_n| \leqslant \exp(-n^{1+1/(\Lambda+\epsilon)}). \tag{7}$$

Let $S_n(z) = \sum_{k=0}^{n} a_k z^k$; then clearly for $0 \leqslant x \leqslant r$,

$$0 \leqslant \frac{1}{S_n(x)} - \frac{1}{f(x)} \leqslant \frac{f(x) - S_n(x)}{f(x)S_n(x)} \leqslant a_0^{-2} \sum_{k=n+1}^{\infty} a_k r^k. \tag{8}$$

From (7) and (8) we get, for $0 \leqslant x \leqslant r$,

$$\begin{aligned} 0 &\leqslant \frac{1}{S_n(x)} - \frac{1}{f(x)} \leqslant a_0^{-2} \sum_{k=n+1}^{\infty} \exp(-k^{1+(1/(\Lambda+\epsilon)} + k \log r) \\ &= a_0^{-2} \sum_{k=n+1}^{\infty} \exp[(-n^{(1/(\Lambda+\epsilon))} + \log r)k]. \end{aligned} \tag{9}$$

Choose

$$2 \log r = n^{(1/(\Lambda+\epsilon))}. \tag{10}$$

Then we have from (9) that

$$\begin{aligned} 0 &\leqslant \frac{1}{S_n(x)} - \frac{1}{f(x)} \leqslant a_0^{-2} \sum_{k=\Lambda+1}^{\infty} \exp\left[\left(\frac{-n^{(1/(\Lambda+\epsilon))}}{2}\right)k\right] \\ &\leqslant C_1 \exp\left(-n^{1+1/(\Lambda+\epsilon)} C_2\right), \end{aligned} \tag{11}$$

where C_1, C_2 are some constants. On the other hand for all $x \geqslant r$, we get

$$0 \leqslant \frac{1}{S_n(x)} - \frac{1}{f(x)} \leqslant \frac{1}{S_n(x)} \leqslant \frac{1}{S_n(r)} = \frac{1}{f(r) - \sum_{k=n+1}^{\infty} a_k r^k}. \tag{12}$$

By assumption

$$0 < \liminf_{r\to\infty} \frac{\log \log M(r)}{\log \log r} = \sigma < \infty.$$

From this we get for all $r \geqslant r_0(\epsilon)$,

$$f(r) = M(r) \geqslant \exp((\log r)^{\sigma-\epsilon}). \tag{13}$$

On the other hand we know from (8) and (11) that

$$\sum_{k=n+1}^{\infty} a_k r^k \leqslant \exp(-C_2 n^{(1/(\Lambda+\epsilon))}). \tag{14}$$

Hence from (12), (13) and (14) we get, with a little manipulation, for all $x \geqslant r$,

$$0 \leqslant \frac{1}{S_n(x)} - \frac{1}{f(x)} \leqslant \frac{1}{S_n(r)} \leqslant \exp\left(-\frac{(\log r)^{\sigma-\epsilon}}{2}\right). \tag{15}$$

From (15) and (10) we obtain for all $x \geqslant r$,

$$0 \leqslant \frac{1}{S_n(x)} - \frac{1}{f(x)} \leqslant \frac{1}{S_n(r)} \leqslant \exp(-C_3 (n)^{(\sigma-\epsilon)/(\Lambda+\epsilon)}), \tag{16}$$

where C_3 is a suitable constant. By choosing ϵ such that $\sigma - \epsilon > \Lambda + \epsilon$, we

get from (11) and (16) the required result, i.e.

$$\lim_{n\to\infty}\left\{\left\|\frac{1}{f(x)}-\frac{1}{P_n(x)}\right\|_{L_\infty[0,\infty)}\right\}^{1/n}=0.$$

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