RATIONAL CHEBYSHEV APPROXIMATION TO
CERTAIN ENTIRE FUNCTIONS OF ZERO
ORDER ON THE POSITIVE REAL AXIS. II

A. R. REDDY

Abstract. A sufficient condition is given on the growth of certain entire functions which guarantee a certain rate of convergence of the error in approximating reciprocals of a class of entire functions by reciprocals of polynomials under the uniform norm on the positive real axis.

Recently we proved

THEOREM 1 [1, Theorem 7]. Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), \( a_0 > 0 \) and \( a_k \geq 0 \) (where \( k \geq 1 \)) be an entire function of zero order satisfying the assumptions that

\[ 1 < \limsup_{r \to \infty} \frac{\log \log M(r)}{\log \log r} = \Lambda + 1 = \rho_l < \infty \]

and

\[ 0 < \liminf_{r \to \infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = b_l, \quad \limsup_{r \to \infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = B_l < \infty, \]

where \( M(r) = \max_{|z| = r} |f(z)| \). Then there exists a sequence of polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) for which

\[ \lim_{n \to \infty} \left\{ \frac{1}{\|f(x) - P_n(x)\|_{L_\infty[0,\infty)}} \right\}^{1/n} = 0. \]

One sees that (2) implies the limit in (1), for suppose, to the contrary,

\[ 0 < \liminf_{r \to \infty} \frac{\log \log M(r)}{\log \log r} = \sigma < \Lambda + 1 \]

\[ = \limsup_{r \to \infty} \frac{\log \log M(r)}{\log \log r} < \infty. \]

Then for each \( \epsilon > 0 \), there is a sequence of values of \( r \) for which

\[ \log M(r) \leq (\log r)^{\sigma + \epsilon}. \]

From (4) we obtain

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\( \frac{(\log M(r))/(\log r)^{\Lambda+1}}{\log r+\epsilon}/(\log r)^{\Lambda+1} \)

and we may choose \( \epsilon \) so that \( \sigma+\epsilon < \Lambda + 1 \). Then clearly we have

\[
\liminf_{r \to \infty} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = b_I = 0,
\]

which contradicts (2).

On the other hand the limit in (1) may not imply (2). For example let

\[
f_1(z) = 1 + \sum_{n=2}^{\infty} \frac{z^n}{\exp[(n \log n)^2]},
\]

\[
f_2(z) = 1 + \sum_{n=2}^{\infty} \frac{z^n}{\exp[(n/\log n)^2]}.
\]

It is easy to calculate for

\[
f_1(z), \quad \Lambda = 1 = \sigma - 1, \quad B_I = 0, \quad f_2(z), \quad \Lambda = 1 = \sigma - 1, \quad B_I = \infty.
\]

Hence it is natural to ask whether one can get (3) under the only assumption

\[
0 < \lim_{r \to \infty} \frac{\log \log M(r)}{\log \log r} = \Lambda + 1 < \infty.
\]

In fact we prove the more general

**Theorem 2.** Let \( f(z) = \sum_{k=0}^{\infty} a_k z^k, a_0 > 0, a_k \neq 0 (k \geq 1) \) be an entire function of zero order satisfying the assumption that

\[
0 < \Lambda < \liminf_{r \to \infty} \frac{\log M(r)}{\log \log r} = \sigma < \infty.
\]

Then there exists a sequence of polynomials \( \{P_n(x)\}_{n=0}^{\infty} \) for which

\[
\lim_{n \to \infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L^p[0,\infty)} \right\}^{1/n} = 0.
\]

**Remark.** The method used here is much simpler than the one used in [1].

**Proof.** By assumption

\[
\limsup_{r \to \infty} \frac{\log M(r)}{\log \log r} = \Lambda + 1.
\]

Then it is known [2, Theorems 1 and 3] that

\[
\limsup_{n \to \infty} \frac{\log n}{\log(n^{-1} \log |a_n|^{-1})} = \Lambda.
\]

From (6) it is easy to get for each \( \epsilon > 0 \) there is an \( n_0 \) depending on \( \epsilon \) such that for all \( n \geq n_0(\epsilon) \),

\[
|a_n| \leq \exp(-n^{1+1/(\Lambda+\epsilon)}).
\]
Let \( S_n(z) = \sum_{k=0}^{n} a_k z^k \); then clearly for \( 0 \leq x \leq r \),

\[
0 \leq \frac{1}{S_n(x)} - \frac{1}{f(x)} \leq \frac{f(x) - S_n(x)}{f(x)S_n(x)} \leq a_0^{-2} \sum_{k=n+1}^{\infty} a_k r^k.
\]

From (7) and (8) we get, for \( 0 \leq x \leq r \),

\[
0 \leq \frac{1}{S_n(x)} - \frac{1}{f(x)} \leq a_0^{-2} \sum_{k=n+1}^{\infty} \exp(-k^{1+1/(\Lambda+\epsilon)} + k \log r) = a_0^{-2} \sum_{k=n+1}^{\infty} \exp\left(\left(-n^{1/(\Lambda+\epsilon)} + \log r\right)k\right).
\]

Choose

\[
2 \log r = n^{1/(\Lambda+\epsilon)}.
\]

Then we have from (9) that

\[
0 \leq \frac{1}{S_n(x)} - \frac{1}{f(x)} \leq a_0^{-2} \sum_{k=\Lambda+1}^{\infty} \exp\left(\left(-n^{1/(\Lambda+\epsilon)} C_2\right) \right)k
\]

where \( C_1, C_2 \) are some constants. On the other hand for all \( x \geq r \), we get

\[
0 \leq \frac{1}{S_n(x)} - \frac{1}{f(x)} \leq \frac{1}{S_n(r)} \leq \frac{1}{f(r) - \sum_{k=n+1}^{\infty} a_k r^k}.
\]

By assumption

\[
0 < \liminf_{r \to \infty} \frac{\log \log M(r)}{\log \log r} = \sigma < \infty.
\]

From this we get for all \( r \geq r_0(\epsilon) \),

\[
f(r) = M(r) \geq \exp((\log r)^{\sigma-\epsilon}).
\]

On the other hand we know from (8) and (11) that

\[
\sum_{k=n+1}^{\infty} a_k r^k \leq \exp(-C_2 n^{1/(\Lambda+\epsilon)}).
\]

Hence from (12), (13) and (14) we get, with a little manipulation, for all \( x \geq r \),

\[
0 \leq \frac{1}{S_n(x)} - \frac{1}{f(x)} \leq \frac{1}{S_n(r)} \leq \exp\left(\left(-\frac{(\log r)^{\sigma-\epsilon}}{2}\right)\right).
\]

From (15) and (10) we obtain for all \( x \geq r \),

\[
0 \leq \frac{1}{S_n(x)} - \frac{1}{f(x)} \leq \frac{1}{S_n(r)} \leq \exp(-C_3 n^{(\sigma-\epsilon)/(\Lambda+\epsilon)}),
\]

where \( C_3 \) is a suitable constant. By choosing \( \epsilon \) such that \( \sigma - \epsilon > \Lambda + \epsilon \), we
get from (11) and (16) the required result, i.e.

$$\lim_{n \to \infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} = 0.$$ 

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References


Department of Mathematics, Michigan State University, East Lansing, Michigan 48823

Current address: Department of Mathematics, Yeshiva University, New York, New York 10033