SUBMANIFOLDS OF A RIEMANNIAN
MANIFOLD WITH
SEMISYMMETRIC METRIC CONNECTIONS

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ABSTRACT. We derive the Gauss curvature equation and the Codazzi-
Mainardi equation with respect to a semisymmetric metric connection on a
Riemannian manifold and the induced one on a submanifold. We then
generalize the theorema egregium of Gauss.

1. Introduction. K. Yano [6] proved that a Riemannian manifold admits a
semisymmetric metric connection with vanishing curvature tensor if and only
if the manifold is conformally flat. Later, T. Imai [3], [4] studied some
properties of hypersurfaces of a Riemannian manifold with a semisymmetric
metric connection, and also obtained the Gauss curvature equation and the
Codazzi-Mainardi equation with respect to a semisymmetric metric connec-
tion on a Riemannian manifold and the induced one on a hypersurface. The
object of this paper is to derive the above two equations with respect to a
semisymmetric metric connection on an \((n + p)\)-dimensional Riemannian
manifold and the induced one on an \(n\)-dimensional submanifold, and also to
generalize the theorema egregium of Gauss. The notation of [5] will be used
for the most part.

2. Gauss equation and Weingarten equation. Let \(M\) be an \(n\)-dimensional
Riemannian manifold isometrically imbedded in an \((n + p)\)-dimensional
Riemannian manifold \(M'\). We denote by \(g\) the Riemannian metric tensor on
\(M'\) as well as the induced one on \(M\). Since \(M\) has codimension \(p\) we can locally
choose \(p\) cross sections \(\xi_i\), \(i = 1, 2, \ldots, p\), of the normal bundle \(T(M')^\perp\) of
\(M\) in \(M'\) which are orthonormal at each point of \(M\).

A linear connection \(\hat{\nabla}'\) on \(M'\) is called a semisymmetric metric connection
if \(\hat{\nabla}'g = 0\) (metric) and the torsion tensor \(\hat{T}'\) of \(\hat{\nabla}'\) satisfies
\(\hat{T}'(X', Y') = \pi(Y')X' - \pi(X')Y'\) (semisymmetric) for \(X', Y' \in \mathfrak{X}(M')\), where \(\pi\) is a 1-
form on \(M'\) [6].

We now assume that a semisymmetric metric connection \(\nabla'\) is given on \(M'\)
by
\[
\hat{\nabla}'_{X'} Y' = \nabla_{X'} Y' + \pi(Y')X' - g(X', Y')P'
\]
for \(X', Y' \in \mathfrak{X}(M')\), where \(\nabla'\) denotes the Riemannian connection with
respect to \(g\) and \(P'\) a vector field on \(M'\) defined by \(g(P', X') = \pi(X')\) for \(X'

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On $M$ we define a vector field $P$ and real-valued functions $\lambda_i, i = 1, 2, \ldots, p$, by decomposing $P'$ into its unique tangential and normal components, thus

$$ P' = P + \sum_{i=1}^{p} \lambda_i \xi_i. $$

If we denote by $\nabla$ the induced Riemannian connection on $M$ from $\nabla'$ on $M'$, then we have the Gauss equation (with respect to $\nabla'$),

$$ \nabla'_{X} Y = \nabla_{X} Y + \sum_{i=1}^{p} h_i(X, Y) \xi_i $$

for $X, Y \in \mathfrak{X}(M)$, where $h_i$ are the second fundamental forms on $M$ [5, pp. 10–21]. Let a connection $\tilde{\nabla}$ on $M$ be induced from the semisymmetric metric connection $\tilde{\nabla}'$ on $M'$ by the equation which may be called the Gauss equation with respect to $\tilde{\nabla}'$,

$$ \tilde{\nabla}'_{X} Y = \tilde{\nabla}_{X} Y + \sum_{i=1}^{p} \hat{h}_i(X, Y) \xi_i $$

for $X, Y \in \mathfrak{X}(M)$, where $\hat{h}_i$ are tensors of type $(0,2)$ on $M$.

From (1), using (2), (3) and (4), we obtain

$$ \tilde{\nabla}_{X} Y + \sum_{i=1}^{p} \hat{h}_i(X, Y) \xi_i = \nabla_{X} Y + \sum_{i=1}^{p} h_i(X, Y) \xi_i $$

$$ + \pi(Y) X - g(X, Y) \left( P + \sum_{i=1}^{p} \lambda_i \xi_i \right), $$

from which we get

$$ \tilde{\nabla}_{X} Y = \nabla_{X} Y + \pi(Y) X - g(X, Y) P $$

for $X, Y \in \mathfrak{X}(M)$, and we also have

$$ \hat{h}_i = h_i - \lambda_i g. $$

Using (6), we get $\tilde{\nabla}_{X} \{ g(Y, Z) \} = (\tilde{\nabla}_{X} g)(Y, Z) + \nabla_{X} \{ g(Y, Z) \}$, from which follows $(\tilde{\nabla}_{X} g)(Y, Z) = 0$ for $X, Y, Z \in \mathfrak{X}(M)$, i.e.,

$$ \tilde{\nabla} g = 0, $$

and

$$ \tilde{T}(X, Y) = T(X, Y) + \pi(Y) X - \pi(X) Y = \pi(Y) X - \pi(X) Y $$

for $X, Y \in \mathfrak{X}(M)$, where $\tilde{T}$ and $T$ denote the torsion tensors of connections $\tilde{\nabla}$ and $\nabla$, respectively. Then from (8) and (9) we have [4]

**Theorem 1.** The induced connection on a submanifold of a Riemannian manifold with a semisymmetric metric connection is also a semisymmetric metric connection.
The Weingarten equation (with respect to $\nabla'$) is given by

\begin{equation}
\nabla'_{\xi_i} = -A_i(X) + D_X \xi_i
\end{equation}

for $X \in \mathfrak{X}(M)$, where $A_i$ are tensors of type (1,1) on $M$ and $D$ is a (metric) connection in the normal bundle $T(M)^\perp$ with respect to the fibre metric induced from $g$ [5, pp. 10-21].

Note that from (3) and (10) we have $h_i(X, Y) = g(Y, A_i(X))$, and thus we get

\begin{equation}
h_i(X, Y) = g(X, A_i(Y)) = g(A_i(X), Y)
\end{equation}

for $X, Y \in \mathfrak{X}(M)$ since $h_i$ are symmetric. While from (1) and (2) we have $\hat{\nabla}'_{\xi_i} = \nabla'_{\xi_i} + \lambda_iX$, which together with (10) implies $\hat{\nabla}'_{\xi_i} = -(A_i - \lambda_iI) \cdot (X) + D_X \xi_i$, where $I$ is the identity tensor. Defining tensors $\hat{A}_i$ of type (1,1) on $M$ by $\hat{A}_i = A_i - \lambda_iI$, we get a more concise expression,

\begin{equation}
\hat{\nabla}'_{\xi_i} = -\hat{A}_i(X) + D_X \xi_i
\end{equation}

for $X \in \mathfrak{X}(M)$, which may be called the Weingarten equation with respect to $\hat{\nabla}'$.

We obtain, using (7) and (11), the following result which will be used later [1], [5, pp. 10-21]:

**Lemma.** The induced linear transformations, also denoted by $A_i$ and $\hat{A}_i$, of the tangent space $T_m(M)$ at $m \in M$ defined by (10) and (12) satisfy, respectively, $h_i(X, Y) = g(A_i(X), Y)$ and $\hat{h}_i(X, Y) = g(\hat{A}_i(X), Y)$ for $X, Y \in T_m(M)$, and thus are symmetric with respect to $g$, i.e., $g(A_i(X), Y) = g(X, A_i(Y))$ and $g(\hat{A}_i(X), Y) = g(X, \hat{A}_i(Y))$ for $X, Y \in T_m(M)$.

The mean curvature normal $H$ of $M$ (with respect to $\nabla$) is given by $H = (1/n) \sum_{j=1}^{p} (trace A_j)\xi_j$ [5, pp. 29-42]. We define similarly the mean curvature normal $\hat{H}$ of $M$ with respect to $\hat{\nabla}$ by $\hat{H} = (1/n) \sum_{j=1}^{p} (trace \hat{A}_j)\xi_j$. Let $X_j, j = 1, 2, \ldots, n$, be $n$ orthonormal local vector fields on $M$. Then $H$ and $\hat{H}$ can be expressed as

\begin{equation}
H = \frac{1}{n} \sum_{i=1}^{p} \left\{ \sum_{j=1}^{n} h_i(X_j, X_j) \right\} \xi_i, \quad \hat{H} = \frac{1}{n} \sum_{i=1}^{p} \left\{ \sum_{j=1}^{n} \hat{h}_i(X_j, X_j) \right\} \xi_i.
\end{equation}

If $h_i = k_i g$, where $k_i$ are real-valued functions on $M$, then $M$ is said to be totally umbilical (with respect to $\nabla$). Similarly, if $\hat{h}_i = k_i g$, then $M$ is said to be totally umbilical with respect to $\hat{\nabla}$ [4], [7, pp. 91-93].

We get from (2), (7) and (13) the following results [4]:

**Theorem 2.** The mean curvature normal of $M$ and that of $M$ with respect to the semisymmetric metric connection $\hat{\nabla}$ coincide if and only if the vector field $P'$ is tangent to $M$.

**Theorem 3.** A submanifold $M$ of a Riemannian manifold $M'$ is totally umbilical if and only if it is totally umbilical with respect to the semisymmetric metric connection $\hat{\nabla}$.
3. Gauss curvature equation and Codazzi-Mainardi equation. We denote by $R'(X', Y')Z' = \nabla_{X'} Y' Z' - \nabla_{Y'} X' Z' - \nabla_{[X', Y']} Z'$ for $X', Y', Z' \in \mathfrak{X}(M')$ and $R(X, Y)Z = \nabla_X Y Z - \nabla_Y X Z - \nabla_{[X, Y]} Z$ for $X, Y, Z \in \mathfrak{X}(M)$ the curvature tensors of $\nabla'$ and $\nabla$, respectively. Then the Gauss curvature equation (with respect to $\nabla'$ and $\nabla$) is given by

$$R'(X, Y)Z = R(X, Y)Z$$

(14)

$$+ \sum_{i=1}^{p} \{ h_i(X, Z) h_i(Y, W) - h_i(Y, Z) h_i(X, W) \}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$, where

$$R'(W', Z', X', Y') = g(R'(X', Y')Z', W').$$

for $X', Y', Z', W' \in \mathfrak{X}(M')$ and

$$R(W, Z, X, Y) = g(R(X, Y)Z, W)$$

for $X, Y, Z, W \in \mathfrak{X}(M)$ are, respectively, the Riemann-Christoffel curvature tensors of $M'$ and $M$ (with respect to $\nabla'$ and $\nabla$), and the Codazzi-Mainardi equation (with respect to $\nabla'$ and $\nabla$) is given by

$$\nabla_i h_j(Y, Z) = g(\nabla_i h_j(Y, Z) D_X \xi_j - \nabla_i h_j(Y, Z) D_Y \xi_j, \xi_j)$$

(16)

for $X, Y, Z \in \mathfrak{X}(M)$.

Next we shall find the Gauss curvature equation and the Codazzi-Mainardi equation with respect to the semisymmetric metric connections $\nabla'$ and $\nabla$. The curvature tensors of $\nabla'$ and $\nabla$ are defined, respectively, by

$$\nabla_i h_j(Y, Z) = g(\nabla_i h_j(Y, Z) D_X \xi_j - \nabla_i h_j(Y, Z) D_Y \xi_j, \xi_j)$$

(15)

$$+ \sum_{j=1}^{p} g((\nabla_j h_i(Y, Z) D_X \xi_j - \nabla_j h_i(Y, Z) D_Y \xi_j, \xi_j)$$

for $X, Y, Z \in \mathfrak{X}(M)$.

We define the Riemann-Christoffel curvature tensors of $M'$ and $M$ with respect to $\nabla'$ and $\nabla$, respectively, by $\hat{R}(W', Z', X', Y') = g(\hat{R}(X', Y')Z', W')$ and $\hat{R}(W, Z, X, Y) = g(\hat{R}(X, Y)Z, W)$ for $X', Y', Z', W' \in \mathfrak{X}(M')$ and $X, Y, Z, W \in \mathfrak{X}(M)$. Then from (16) and the Lemma, we obtain the Gauss curvature equation with respect to $\nabla'$ and $\nabla$ [4]:
\[ \hat{R}(W, Z, X, Y) = \hat{R}(W, Z, X, Y) \]

(17) \[ \sum_{i=1}^{p} \left\{ h_i(X, Z)h_i(Y, W) - h_i(Y, Z)h_i(X, W) \right\} \]

for \( X, Y, Z, W \in \mathfrak{X}(M) \). We also have from (16) the Codazzi-Mainardi equation with respect to \( \hat{\nabla}' \) and \( \hat{\nabla} \) [4]:

\[ \hat{R}'(\xi_i, Z, X, Y) = (\hat{\nabla}_{\xi_i} h_i) (Y, Z) - (\hat{\nabla}_{\xi_i} h_i) (X, Z) + h_i(\pi(Y)X - \pi(X)Y, Z) \]

(18) \[ + \sum_{j=1}^{p} g((\hat{h}_j(Y, Z)D_{\xi_j} - \hat{h}_j(X, Z)D_{\xi_j} \xi_j, \xi_i) \]

for \( X, Y, Z \in \mathfrak{X}(M) \).

Now we suppose the Riemannian manifold \( M' \) is conformally flat and that the submanifold \( M \) is totally umbilical, then we can assume \( \hat{R}' = 0 \) [6], and we also have \( h_i = k_i g \), since \( M \) is totally umbilical with respect to \( \hat{\nabla} \) by Theorem 3. Then from (17) we get

(19) \[ \hat{R}(W, Z, X, Y) = \left\{ g(Y, Z)g(X, W) - g(X, Z)g(Y, W) \right\} \sum_{i=1}^{p} (k_i)^2 \]

for \( X, Y, Z, W \in \mathfrak{X}(M) \), which implies that \( M \) is also conformally flat \( (n > 3) \) [3]. Thus we have [4]

**Theorem 4.** A totally umbilical submanifold in a conformally flat Riemannian manifold is conformally flat.

4. **Theorema egregium.** We obtain a generalization of the theorema egregium of Gauss with respect to semisymmetric metric connection by the method of N. Hicks [1].

From the Gauss curvature equation (17) and the Lemma, we get

\[ \hat{R}'(X, Y, X, Y) = \hat{R}(X, Y, X, Y) \]

(18) \[ + \sum_{i=1}^{p} \left\{ g(\hat{A}_i(X), Y)^2 - g(\hat{A}_i(X), X)g(\hat{A}_i(Y), Y) \right\} \]

for \( X, Y \in \mathfrak{X}(M) \). Therefore we have

**Theorem 5.** Let \( \mathfrak{P} \) be a 2-dimensional subspace of \( T_m(M) \), and let \( \hat{R}'(\mathfrak{P}) \) and \( \hat{R}(\mathfrak{P}) \) be, respectively, the sectional curvatures of \( \mathfrak{P} \) in \( M' \) and \( M \) with respect to the semisymmetric connections \( \hat{\nabla}' \) and \( \hat{\nabla} \). If \( X \) and \( Y \) form an orthonormal base of \( \mathfrak{P} \), then

(20) \[ \hat{R}'(\mathfrak{P}) = \hat{R}(\mathfrak{P}) + \sum_{i=1}^{p} \left\{ g(\hat{A}_i(X), Y)^2 - g(\hat{A}_i(X), X)g(\hat{A}_i(Y), Y) \right\}. \]

As immediate consequences of Theorem 5 we get [6]

**Corollary 1.** If \( \dim M' = 3 \) and \( M \) is a surface in \( M' \), then the determinant of \( \hat{A}_i \) (where there is now only one such map) is independent of \( \hat{A}_i \) but depends only on the Riemannian metric tensor \( g \) and the semisymmetric metric connections \( \hat{\nabla}' \) and \( \hat{\nabla} \).
Corollary 2. If $M'$ is a conformally flat Riemannian manifold of dimension 3 and $M$ is a surface in $M'$, then there exists a semisymmetric metric connection $\hat{\nabla}$ on $M$ for which $\det \hat{A}_i$ is an intrinsic invariant of $M$, and, when $P'$ is tangent to $M$, $\det \hat{A}_i (= \hat{\mathcal{R}}(\hat{\otimes}))$ is equal to $\det A_i$ which is the Gaussian curvature of $M$.

References

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