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UNITARIES AND PARTIAL ISOMETRIES IN A REAL W^*-ALGEBRA

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ABSTRACT. The group of unitaries in a real W^*-algebra without a finite type I direct summand is connected. This fact is used to characterize the components of the set of partial isometries in such algebras.

1. **Introduction.** Many theorems for complex W^*-algebras apply equally well to real W^*-algebras, often with identical proofs, while others, such as the spectral theorem do not. However, we show in §2 that one of the standard consequences of the spectral theorem, the connectivity of the unitary operators in a complex W^*-algebra, holds for real W^*-algebras without a finite type I summand. In §3 we use the result of §2 to extend to real W^*-algebras without a finite type I summand, and to all complex W^*-algebras, a result of Halmos and McLaughlin [4] characterizing the components of partial isometries.

For any real, separable, infinite-dimensional Hilbert space $\mathcal{H}$ let $\mathcal{H}_c = \mathcal{H} \otimes \mathbf{C}$ be its complexification. We may identify $\mathcal{H}_c$ with $\mathcal{H} \oplus \mathcal{H}$ (as real spaces), and, if A is a bounded (real) linear operator on $\mathcal{H}$, we may identify A with the (complex) linear operator on $\mathcal{H}_c$ whose matrix on $\mathcal{H} \oplus \mathcal{H}$ is $\left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Since multiplication by i on $\mathcal{H}_c$ corresponds to the matrix $\left(\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right)$, B is a bounded (complex) linear operator on $\mathcal{H}_c$ if and only if B is of the form $B_1 + iB_2$, where B_1 and B_2 are bounded real linear operators on $\mathcal{H}$, i.e., matricially B has the form

$$\begin{pmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{pmatrix}.$$

If $\mathfrak{A}$ is a real W^*-algebra of operators on $\mathcal{H}$, let $\mathfrak{A}_c$ denote its complexification (that is, $\mathfrak{A}_c$ is just the set of all $A_1 + iA_2$, where A_1 and A_2 are in $\mathfrak{A}$). With the canonical identifications above, $\mathfrak{A}_c$ is a complex W^*-algebra of operators on $\mathcal{H}_c$.

2. The group of unitaries.

THEOREM 1. *The unitary group in a real W^*-algebra $\mathfrak{A}$ without a finite type I direct summand is arcwise connected.*

PROOF. Let U be a unitary element of $\mathfrak{A}$. Then U is also unitary in $\mathfrak{A}_c$ so we may use the spectral theorem to conclude $U = \exp(iH)$ for some Hermitian H in $\mathfrak{A}_c$. Following Martin [6], we assume without loss of generality that H is "reduced" (i.e., that $\sigma(H) \subset [0, 2\pi]$ and 2π is not in the point spectrum of H).

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We may decompose H into $\pi B + iS$, where B is a selfadjoint element of $\mathfrak{A}$ and S is a skew-adjoint element of $\mathfrak{A}$. Since U is actually in $\mathfrak{A}$ (hence, orthogonal), we may use a result of Martin [6, p. 602] to conclude that B is idempotent and commutes with S. Thus, we have

$$U = \exp(iH) = \exp(i\pi B - S) = \exp(i\pi B)\exp(-S).$$

Note that B is actually a projection and that an easy calculation shows $\exp(i\pi B) = I - 2B$.

What we wish to do now is to find some skew-adjoint T in $\mathfrak{A}$ such that $\exp(\pi T) = I - 2B = \exp(i\pi B)$. Then if we set U_t to be $\exp(t\pi T)\exp(-tS)$ for $0 \leqslant t \leqslant 1$, each U_t lies in $\mathfrak{A}$ and is unitary in $\mathfrak{A}_c$, so we have constructed a continuous path of unitaries in $\mathfrak{A}$ joining U to I. This, of course, will prove the theorem.

The operator T is, roughly speaking, a square root of $-I$ on $B\mathcal{H}$ and zero on $(I - B)\mathcal{H}$. More precisely, we may apply Theorems 45 and 49 of Kaplansky [5] and the standard direct sum decomposition of $\mathfrak{A}$ to conclude that $B = Q_1 + Q_2$, where Q_1 and Q_2 are equivalent orthogonal projections in $\mathfrak{A}$. Thus, there is a partial isometry V of $\mathfrak{A}$ with $V^*V = Q_1$, and $VV^* = Q_2$. We set T to be $V - V^*$. Then T is skew-adjoint, commutes with B, and satisfies $BTB = T$. Thus, if we consider B and T as elements of $\mathfrak{A}_c$, T is an element of the W^*-algebra $B\mathfrak{A}_c B$. Then $R = -iT$ is a selfadjoint unitary in $B\mathfrak{A}_c B$, so its spectrum satisfies $\sigma(R) \subset \{+1, -1\}$. For $\lambda \neq 0, +1, -1$, let C_λ be the inverse of $(R - \lambda)$ in $B\mathfrak{A}_c B$. Then $(I - B)(-\lambda^{-1})(I - B) + BC_\lambda B$ is the inverse of $(R - \lambda)$ in $\mathfrak{A}_c$, so the spectrum of R in $\mathfrak{A}_c$ is contained in $\{0, +1, -1\}$. Let E be the spectral measure on $\sigma(R)$ for the spectral decomposition of R. Then $E(\{+1\})$, $E(\{-1\})$, and $E(\{0\})$ are projections in $\mathfrak{A}_c$ such that

$$E(\{+1\}) + E(\{-1\}) = B$$

and, for a function f on $\sigma(R)$,

$$f(R) = f(+1)E(\{+1\}) + f(-1)E(\{-1\}) + f(0)E(\{0\}).$$

Setting $f(\lambda) = \exp(i\pi\lambda)$ yields

$$\exp(i\pi R) = -E(\{+1\}) - E(\{-1\}) + E(\{0\}) = -B + (I - B) = I - 2B.$$

But $i\pi R$ is just πT, so T is the operator for which we were searching. Q.E.D.

3. **Partial isometries.** If E and F are two projections of a W^*-algebra $\mathfrak{A}$, write $E \sim F$ if E and F are equivalent in the usual way, i.e., if there is a partial isometry W of $\mathfrak{A}$ with $W^*W = E$ and $WW^* = F$. If W and V are any two partial isometries of $\mathfrak{A}$, let $E = W^*W$, $E' = WW^*$, $F = V^*V$, and $F' = VV^*$. Write $W \approx V$ if $(I - E) \sim (I - F)$, $E' \sim F'$, $(I - E') \sim (I - F')$. (Note that this implies $E \sim F$ also, since $E \sim E'$ and $F \sim F'$.)

Lemma 1. *Let W and V be partial isometries in a W^*-algebra $\mathfrak{A}$ (real or complex). If $\|W - V\| < \frac{1}{2}$, then $W \approx V$.*

Proof. Using the notation of the previous paragraph, we have

$$\|E - F\| = \|W^*W - V^*V\| \leqslant \|W^*\| \, \|W - V\| + \|W^* - V^*\| \, \|V\|$$
$$\leqslant 2\|W - V\| < 1.$$

Similarly, $\|E' - F'\| < 1$. The polar decomposition theorem holds for $\mathfrak{A}$ [5, Theorem 65], so we may apply Lemma 1.2 of Araki, Smith, and Smith [1] to conclude $(I - E) \sim (I - F)$, $E' \sim F'$, and $(I - E') \sim (I - F')$. Q.E.D.

In the opposite direction, we have the following result.

LEMMA 2. *Let $\mathfrak{A}$ be a W^*-algebra in which the unitaries are arcwise connected. Let W and V be two partial isometries in $\mathfrak{A}$ such that $W \approx V$. Then there exists a continuous path W_t* $(0 \leqslant t \leqslant 1)$ *of partial isometries of $\mathfrak{A}$ such that $W_0 = W$, $W_1 = V$, and $W_t \approx W$ for each t.*

Note that this lemma extends the result of Halmos and McLaughlin [4] for the case that $\mathfrak{A}$ is the algebra of all bounded operators on a complex Hilbert space. In fact, the proof is practically identical to the proof (of Douglas) for that special case (see Halmos [3]) and will be omitted. (One need only note that the assumptions of the lemma suffice to insure that the partial isometries of Douglas' proof do lie in the W^*-algebra $\mathfrak{A}$.)

Combining Lemmas 1 and 2 we have the following result.

THEOREM 2. *Let $\mathfrak{A}$ be a W^*-algebra whose unitary group is arcwise connected. Then two partial isometries W and V of $\mathfrak{A}$ are in the same component of the set of partial isometries if and only if $W \approx V$.*

COROLLARY . *If W and V are two partial isometries in a real W^*-algebra with no finite type* I *direct summand, then W and V are in the same component of the set of partial isometries if and only if $W \approx V$.*

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