ON BLASCHKE PRODUCTS DIVERGING EVERYWHERE ON THE BOUNDARY OF THE UNIT DISC

C. N. LINDEN

Abstract. If the moduli of the zeros of a Blaschke product increase sufficiently slowly the arguments of the zeros may be so chosen that the product diverges everywhere on \( \{z: |z| = 1\} \).

A sequence of complex numbers \( \{a_n\} \) in the unit disc is a Blaschke sequence if and only if \( \sum_n (1 - |a_n|) < \infty \). If the first \( m \) terms of \( \{a_n\} \) are zero and the remaining terms are nonzero, then the corresponding Blaschke product is defined by the formula

\[
B(z, \{a_n\}) = z^m \prod_{n > m} b(z, a_n) = z^m \prod_{n > m} \frac{\overline{a_n}(a_n - z)}{|a_n|(1 - \overline{a_n}z)}.
\]

The Blaschke product converges absolutely for \( |z| < 1 \), and, if

\[
\sum_n (1 - |a_n|) \log \frac{1}{1 - |a_n|} < \infty
\]

converges, Frostman [1] has shown that it also converges absolutely almost everywhere on \( \{z: |z| = 1\} \). It will be shown here that if (1) does not converge then the arguments of the terms of the sequence \( \{a_n\} \) can be so chosen that \( B(z, \{a_n\}) \) diverges whenever \( |z| = 1 \). Although I am not aware of the existence, in the published literature, of examples of Blaschke products which diverge everywhere on \( \{z: |z| = 1\} \), some unpublished examples have been obtained by Clunie and Piranian.

Theorem. Let \( \{r_n\} \) be a positive Blaschke sequence such that

\[
\sum_{n=1}^{\infty} (1 - r_n) \log(1/(1 - r_n))
\]

diverges, and let

\[
\theta_N = \sum_{n=1}^{N} (1 - r_n) \log \frac{1}{1 - r_n} \quad (N = 1, 2, 3, \ldots).
\]

Then if \( a_n = r_n e^{i\theta_n} \) the Blaschke product \( B(z, \{a_n\}) \) diverges for \( z = e^{i\theta} \) \( (0 \leq \theta < 2\pi) \).
For each natural number \( k \) we choose \( N (= N(k)) \) to satisfy the inequality
\[
\theta_{N-1} \leq \theta + 2k\pi < \theta_N,
\]
and, since \( \theta_n - \theta_{n-1} \leq 1/e \), we can then choose \( M (= M(N)) \) so that
\[
\frac{1}{2} < \theta_N - \theta_M < 1.
\]
Clearly \( \{N(k)\} \) and \( \{M(N(k))\} \) are unbounded increasing sequences.

Elementary properties of the Blaschke factors show that
\[
b(e^{i\theta}, a_n) = \left( \frac{r_n e^{i(\pi - \theta + \theta_n)} + e^{-i(\pi - \theta + \theta_n)}}{|1 - \overline{a_n} e^{i\theta}|} \right)^2 = e^{-2i\varphi_n},
\]
where
\[
\varphi_n = \arctan \frac{1 - r_n}{(1 + r_n)\tan \frac{1}{2}(\theta - \theta_n)}.
\]

We need to show that \( \sum_{n=1}^{\infty} \varphi_n \) diverges.

We consider \( \varphi_n \) on the set of integers \( n \) that satisfy \( M \leq n < N \). For such \( n \) we have
\[
\tan \frac{1}{2}(\theta - \theta_n) < \tan \frac{1}{2}(\theta_N - \theta_n) < \theta_N - \theta_n
\]
since \( 0 < \theta_N - \theta_n < 1 \), whence
\[
\varphi_n > \arctan \frac{1 - r_n}{2(\theta_N - \theta_n)}.
\]

If
\[
(1 - r_n)/(\theta_N - \theta_n) > 2,
\]
then \( \varphi_n > \pi/4 \). On the other hand, if (3) does not hold then
\[
\varphi_n > \frac{1 - r_n}{4(\theta_N - \theta_n)} = \frac{\theta_n - \theta_{n-1}}{4 \log(1/(1 - r_n))(\theta_N - \theta_n)}
\]
\[
> \frac{1}{4 \log(1/(1 - r_n))} \log \left( 1 + \frac{\theta_n - \theta_{n-1}}{\theta_N - \theta_n} \right)
\]
\[
= \frac{\log(1/(\theta_N - \theta_n)) - \log(1/(\theta_N - \theta_{n-1}))}{4 \log(1/(1 - r_n))}
\]
\[
\geq \frac{\log(1/(\theta_N - \theta_n)) - \log(1/(\theta_N - \theta_{n-1}))}{4 \log(1/(1 - r_{n+1}))}.
\]

Thus we have \( \sum_{n=M}^{N-1} \varphi_n > \pi/4 \) if (3) holds for any integer \( n \) in \([M, N - 1]\).

Otherwise we have
\[
\sum_{n=M}^{N-1} \varphi_n > \frac{\log(1/(\theta_N - \theta_{N-1})) - \log(1/(\theta_N - \theta_{M-1}))}{4 \log(1/(1 - r_N))} - \frac{\log(1/(1 - r_M))}{4 \log(1/(1 - r_M))}
\]
\[
> \frac{1}{4} \log \frac{\log(1/(1 - r_N))}{\log(1/(1 - r_M))} - \frac{\log 2}{4 \log(1/(1 - r_M))}.
\]
since $\theta_N - \theta_{M-1} > \frac{1}{2}$ by (2). Since $\lim_{n \to \infty} r_n = 1$, it follows that $\sum_{n=1}^{\infty} \varphi_n$ diverges.

**Reference**


*University College of Swansea, Singleton Park, Swansea, Wales, United Kingdom*