ON A THEOREM OF BRICKMAN-FILLMORE

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Abstract. Let \( V \) be a finite dimensional vector space over an arbitrary field. We show that if \( \dim V \leq 3 \) and if \( A, B \) and \( C \) are pairwise commuting linear transformations on \( V \) such that every subspace invariant for both \( A \) and \( B \) is also invariant for \( C \), then \( C \) is a polynomial in \( A \) and \( B \). (Brickman and Fillmore proved that if \( B = 0 \) then this statement is true for any finite dimensional vector space \( V \).) An example shows that this is not true for \( \dim V > 3 \).

In [1] L. Brickman and P. A. Fillmore proved that if \( A \) and \( B \) are commuting linear transformations on a finite dimensional vector space over an arbitrary field and if every subspace invariant for \( A \) is also invariant for \( B \), then \( B \) is a polynomial in \( A \). Peter Fillmore suggested the following question (conveyed to me by Constantin Apostol):

If \( A, B \) and \( C \) are pairwise commuting linear transformations on a finite dimensional vector space \( V \) over an arbitrary field and if every subspace invariant for both \( A \) and \( B \) is also invariant for \( C \), then is \( C \) a polynomial in \( A \) and \( B \)?

We shall prove that the answer to this question is true if the dimension of \( V \) is not more than 3 and false otherwise.

Suppose the dimension of \( V \) is 2. If \( A \) has no nontrivial invariant subspace then \( C \) is a polynomial in \( A \) by the Brickman-Fillmore result. If \( A \) is a scalar multiple of the identity then \( C \) is a polynomial in \( B \). Similar statements can also be made for \( B \). Finally, if \( A \) has a 1-dimensional eigenspace then \( A, B \) and \( C \) can be represented by upper triangular matrices relative to a fixed basis. By subtracting appropriate scalar multiples of the identity from \( A, B, \) and \( C \), we may assume that:

\[
A = \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & c_1 \\ 0 & c_2 \end{pmatrix}.
\]

Since \( A \) and \( C \) commute we have \( a_1 c_2 = c_1 a_2 \). Thus (i) \( a_1 \neq 0 \) implies \((c_1/a_1)A = C\), (ii) \( a_2 \neq 0 \) implies \((c_2/a_2)A = C \) and (iii) \( a_1 = a_2 = 0 \) implies \( C \) is a polynomial in \( B \).

The proof for the case when the dimension of \( V \) is 3 is obtained by considering the possible representations of \( A \) given by the rational decomposition theorem. We omit the details.

Finally, let
An easy computation shows that

\[ AB = BA = AC = CA = BC = CB = 0 \quad \text{and} \quad A^2 = B^2 = C^2 = 0. \]

It follows from these that \( C \) is not a polynomial in \( A \) and \( B \). To show that every subspace invariant under \( A \) and \( B \) is also invariant under \( C \) it is sufficient to consider cyclic subspaces (that is, subspaces generated by the action of \( A \) and \( B \) on a single vector). An easy calculation shows that if \( x \) is any vector, then \( Cx \) is a linear combination of \( Ax \) and \( Bx \). This example can be extended to the case \( \dim V > 4 \) via direct sums.

**Reference**


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