ON $H$-CLOSED SPACES

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Abstract. A characterization of $H$-closed spaces in terms of projections is given along with relating properties.

Introduction. The primary purpose of this paper is to give a characterization of $H$-closed spaces which is an analogue to the following theorem for compact spaces: A space $X$ is compact if and only if the projection from $X \times Y$ onto $Y$ is a closed function for every space $Y$ [9, p. 21].

Following the notation of [6], we utilize the notion of $\theta$-closed subsets of a topological space from [11, p. 103] and our characterization is stated as follows:

Theorem. A Hausdorff space $X$ is $H$-closed if and only if for every space $Y$, the projection from $X \times Y$ onto $Y$ takes $\theta$-closed subsets onto $\theta$-closed subsets.

Throughout, cl$(K)$ will denote the closure of a set $K$.

Preliminary definitions and theorems.

Definition 1. A net in a topological space is said to $\theta$-converge ($\theta$-accumulate) [6, Definition 3] to a point $x$ in the space if the net is eventually (frequently) in cl$(V)$ for each $V$ open about $x$.

Definition 2. A point $x$ in a topological space $X$ is in the $\theta$-closure [11, p. 103] of a set $K \subset X$ ($\theta$-cl$(K)$) if cl$(V) \cap K \neq \emptyset$ for any $V$ open about $x$.

Definition 3. A subset $K$ of a topological space is $\theta$-closed [11, p. 103] if it contains its $\theta$-closure (i.e., $\theta$-cl$(K) \subset K$).

The following theorems give some parallels of properties of closure and closed sets in a topological space for $\theta$-closure and $\theta$-closed sets in the space and some relationships between these notions. The proofs of these theorems are straightforward and are omitted [11, Lemmas 1, 2, 3].

Theorem 1. A point $x$ in a topological space is in the $\theta$-closure of a subset $K$ if and only if there is a net $x_{\alpha}$ in $K$ which $\theta$-converges to $x$ ($x_{\alpha} \theta \rightarrow x$).

Theorem 2. In any topological space
(a) the empty set and the whole space are $\theta$-closed,
(b) arbitrary intersections and finite unions of $\theta$-closed sets are $\theta$-closed,
(c) cl$(K) \subset \theta$-cl$(K)$ for each subset $K$,
(d) a $\theta$-closed subset is closed.
Example 1. Each nonempty countable subset of the set of reals endowed with the co-countable topology is closed but not $\theta$-closed.

Main results. There are several characterizations of $H$-closed spaces in the literature [2, p. 145], [1, p. 97]. For a definition, we use the following:

Definition 4. A Hausdorff space $X$ is $H$-closed if every open cover $\mathcal{U}$ of $X$ contains a finite subcollection $\mathcal{V}$ such that $\{\text{cl}(V): V \in \mathcal{V}\}$ covers $X$.

We also make use of the following theorem immediately gotten from [11, Theorem 2]:

Theorem 3. A Hausdorff space is $H$-closed if and only if each net in the space has a $\theta$-convergent subnet.

Definition 5. A function $g: X \to Y$ is weakly continuous [6, Theorem 6] if for each net $x_{\alpha}$ in $X$ such that $x_{\alpha} \to x$, the net $g(x_{\alpha}) \to g(x)$.

Definition 6. A function $g: X \to Y$ has a strongly-closed graph [6, p. 473] if for each $(x, y) \in (X \times Y) - G(g)$, there are open sets $U$ and $V$ about $x$ and $y$, respectively, such that $(U \times \text{cl}(V)) \cap G(g) = \emptyset$.

It is known that a function $g: X \to Y$ has a closed graph if and only if whenever a net $x_{\alpha} \to x$ in $X$ and $g(x_{\alpha}) \to y$ in $Y$, it follows that $g(x) = y$ [13, p. 115]. We have the following similar result for functions with strongly-closed graphs.

Theorem 4. A function $g: X \to Y$ has a strongly-closed graph if and only if whenever a net $x_{\alpha} \to x$ in $X$ and $g(x_{\alpha}) \to y$ in $Y$, it follows that $g(x) = y$.

Proof. Let $g$ have a strongly-closed graph and let $x_{\alpha}$ be a net in $X$ satisfying $x_{\alpha} \to x$ and $g(x_{\alpha}) \to y$. Then $(V \times \text{cl}(W)) \cap G(g) \neq \emptyset$ for $V, W$ open about $x$ and $y$ respectively. So, $(x, y) \in G(g)$ and $g(x) = y$. For the converse, let $(x, y) \in (X \times Y) - G(g)$. Then $y \neq g(x)$, and there is no net $x_{\alpha}$ in $X$ satisfying $x_{\alpha} \to x$ and $g(x_{\alpha}) \to y$. If $(V_{\alpha} \times \text{cl}(W_{\alpha})) \cap G(g) = \emptyset$ for each pair $V_{\alpha}, W_{\alpha}$ of sets open about $x$ and $y$ respectively, choose $(x_{\alpha, \xi}, g(x_{\alpha, \xi})) \in (V_{\alpha} \times \text{cl}(W_{\alpha})) \cap G(g)$. The ordering of $(V_{\alpha} \times \text{cl}(W_{\alpha}))$ by inclusion renders $(x_{\alpha, \xi}, g(x_{\alpha, \xi}))$ a net with $x_{\alpha, \xi} \to x$ and $g(x_{\alpha, \xi}) \to y$, a contradiction. Therefore, there are sets $V, W$ open about $x, y$ respectively, and satisfying $(V \times \text{cl}(W)) \cap G(g) = \emptyset$: and $G(g)$ is strongly-closed. This completes the proof.

We may use the characterizations above to give a proof of the following theorem which is different and shorter than that given in [6]. If $(x_{\alpha}, D)$ is a net in a space $X$, we will denote $\{x_{\alpha}: \alpha > \mu\}$ by $T_{\mu}$ for each $\mu \in D$. Using this notation it is clear that $x_{\alpha} \theta$-converges ($\theta$-accumulates) to a point $x \in X$ if for each open $V$ about $x$, there is a $\mu \in D$ satisfying (each $\mu \in D$ satisfies) $T_{\mu} \subseteq \text{cl}(V)$ $(T_{\mu} \cap \text{cl}(V) \neq \emptyset)$. Let $\mathcal{S}$ denote a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces.

Theorem 5. A Hausdorff space $Y$ is $H$-closed if and only if for every space in class $\mathcal{S}$, each $g: X \to Y$ with a strongly-closed graph is weakly continuous.

Proof. Let $Y$ be $H$-closed, let $X$ be any space and let $g: X \to Y$ have a strongly-closed graph. Let $x_{\alpha} \to x$ in $X$. Then $g(x_{\alpha})$ is a net in $Y$, so there is a
subnet \( y_\beta \) of \( x_\alpha \) and \( y \in Y \) with \( g(y_\beta) \to y \). By Theorem 4, \( g(x) = y \). Let \( V \) be a regular open set about \( g(x) \). If \( g(x_\alpha) \) is not eventually in \( \text{cl}(V) \), there is a subnet \( z_\mu \) of \( x_\alpha \) such that \( g(z_\mu) \) \( \theta \)-converges to some \( z \in Y - V \) since \( Y - V \) is a regular closed set and thus \( H \)-closed. This then forces \( g(x) \in Y - V \), a contradiction. So \( g(x_\alpha) \to g(x) \). For the converse, let \( x_0 \in Y \) and let \( (x_\alpha, D) \) be a net in \( Y - \{x_0\} \) with no \( \theta \)-accumulation point in \( Y - \{x_0\} \). Let \( X = \{x_\alpha: \alpha \in D\} \cup \{x_0\} \) with the topology generated by \( \{\{x_\alpha\}: \alpha \in D\} \) and \( \{T_\mu \cup \{x_0\}: \mu \in D\} \) as the basic open sets. \( X \) is a Hausdorff door space [7, p. 76] and is easily shown to be in class \( \mathbb{S} \). Let \( i: X \to Y \) be the identity function and let \( (x, y) \in (X \times Y) - G(i) \). If \( x \neq x_0 \), then \( \{x\} \) is open in \( X \); choose \( V \) open in \( Y \) about \( y \) with \( x \in \text{cl}(V) \). Then, clearly, \( (\{x\} \times \text{cl}(V)) \cap G(i) = \emptyset \). If \( x = x_0 \), then \( y \neq x_0 \); so there is a \( \mu \in D \) and a \( V \) open in \( Y \) about \( y \) satisfying \( x_0 \notin \text{cl}(V) \) and \( T_\mu \cap \text{cl}(V) = \emptyset \). So \( X - \text{cl}(V) \) is open in \( X \) about \( x \) and \( [(X - \text{cl}(V)) \times \text{cl}(V)] \cap G(i) = \emptyset \). Thus, \( i \) has a strongly-closed graph and is weakly continuous. Consequently, if \( V \) is open about \( x_0 \), there is a \( \mu \in D \) satisfying \( T_\mu \subset \text{cl}(V) \) [8, p. 44], so \( x_\alpha \to x_0 \). This completes the proof.

In [6, p. 474], an example is given to show that the strongly-closed graph condition in Theorem 5 cannot be relaxed to a closed graph condition. This example was extracted from [12] and is not described explicitly in [6] presumably because of its somewhat complicated description. We now exhibit a space with a simpler description which meets the purposes of the example in [6].

**Example 2.** Let \( N \) be the set of positive integers and let \( X = \{0\} \cup [1, \infty) \) with the topology generated by the usual subspace topology of the reals on \([1, \infty)\) and \( \{\{0\} \cup \bigcup_{m \in \mathbb{N}} (k, k + 1): m \in \mathbb{N}\} \) as basic open sets.

(a) The space \( X \) is Hausdorff.

(b) The space \( X \) is not compact since \( N \) is an infinite subset of \( X \) without accumulation points.

(c) The space \( X \) is \( H \)-closed.

(d) The function \( g \), from \( \{1, 1/n: n \in \mathbb{N}\} \cup \{1\} \) with the subspace topology, defined by \( g(1) = 1 \) and \( g(1 + 1/n) = n \) for each \( n \in \mathbb{N} \) has a closed graph which is not strongly-closed. Also, \( g \) is not weakly continuous at \( 1 \).

In [3], [4], [5], and [10], theorems of the following form are proved; \( X \) has property \( \lambda \) if and only if the projection \( \pi_x: X \times Y \to Y \) is closed for each space \( Y \) in a certain class. The next four theorems and main results give an analogue of this form for \( H \)-closed spaces.

**Theorem 6.** If \( X \) is an \( H \)-closed space then the projection from \( X \times Y \) onto \( Y \) takes \( \theta \)-closed subsets onto \( \theta \)-closed subsets for any space \( Y \).

**Proof.** Let \( X \) be \( H \)-closed, let \( Y \) be any space and let \( K \subset X \times Y \) be \( \theta \)-closed. Let \( y \in \theta-\text{cl}(\pi_x(K)) \). There is a net \( (x_\alpha, y_\alpha) \in K \) with \( y_\alpha \to y \). There is a subnet \( x_\alpha^\gamma \) of \( x_\alpha \) and \( x \in X \) with \( x_\alpha^\gamma \to x \). So \( (x_\alpha^\gamma, y_\alpha^\gamma) \to (x, y) \) and \( (x, y) \in \theta-\text{cl}(K) \subset K \). Thus \( y \in \pi_x(K) \).

**Theorem 7.** If \( X \) is a Hausdorff space and the projection from \( X \times Y \) onto \( Y \)
takes \( \theta \)-closed subsets onto closed subsets for every space \( Y \), then \( X \) is H-closed.

**Proof.** Let \( (y_\alpha, D) \) be a net in \( X \) with no \( \theta \)-convergent subnet and let \( y_0 \not\in X \). Let \( Y = \{y_\alpha: \alpha \in D\} \cup \{y_0\} \) with the topology generated by \( \{(y_\alpha): \alpha \in D\} \) and \( \{T_\mu \cup \{y_0\}: \mu \in D\} \) as basic open sets. Let \( K = \{(y_\alpha, y_\alpha): \alpha \in D\} \) and let \((a, b) \in (X \times Y) - K\). Then \( a \neq y_0 \) and \( a \neq b \). Let \( V \) be open about \( a \) satisfying \( \{b, y_0\} \subset Y - \text{cl}(V) \) and \( T_\mu \subset Y - \text{cl}(V) \) for some \( \mu \in D \). Then \( Y - \text{cl}(V) \) is open and closed in \( Y \) and so \( X \times (Y - \text{cl}(V)) \) is open about \((a, b)\). Also,

\[
\text{cl}[V \times (Y - \text{cl}(V))] \cap K = (\text{cl}(V) \times (Y - \text{cl}(V))) \cap K = \emptyset.
\]

Thus, \((a, b) \not\in \theta\text{-cl}(K)\) and \( K \) is \( \theta \)-closed. \( \pi_y(K) \) is therefore closed in \( Y \) and \( y_0 \not\in \text{cl}(\pi_y(K)) \). This is a contradiction establishing the result.

Combining Theorems 6 and 7, we get the promised result.

**Theorem 8.** A Hausdorff space \( X \) is H-closed if and only if for every space \( Y \), the projection from \( X \times Y \) onto \( Y \) takes \( \theta \)-closed subsets onto \( \theta \)-closed subsets.

Noting that the space \( Y \) used in the proof of Theorem 7 is a Hausdorff door space and is in the class \( S \) whose description precedes Theorem 5, we may state the following theorem.

**Theorem 9.** A Hausdorff space \( X \) is H-closed if and only if for every Hausdorff door space \( (space \in class \( S \)) \( Y \), the projection from \( X \times Y \) onto \( Y \) takes \( \theta \)-closed subsets onto \( \theta \)-closed subsets.

**References**


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