ON H-CLOSED SPACES

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Abstract. A characterization of H-closed spaces in terms of projections is given along with relating properties.

Introduction. The primary purpose of this paper is to give a characterization of H-closed spaces which is an analogue to the following theorem for compact spaces: A space $X$ is compact if and only if the projection from $X \times Y$ onto $Y$ is a closed function for every space $Y$ [9, p. 21].

Following the notation of [6], we utilize the notion of $\theta$-closed subsets of a topological space from [11, p. 103] and our characterization is stated as follows:

Theorem. A Hausdorff space $X$ is H-closed if and only if for every space $Y$, the projection from $X \times Y$ onto $Y$ takes $\theta$-closed subsets onto $\theta$-closed subsets.

Throughout, $\text{cl}(K)$ will denote the closure of a set $K$.

Preliminary definitions and theorems.

Definition 1. A net in a topological space is said to $\theta$-converge ($\theta$-accumulate) [6, Definition 3] to a point $x$ in the space if the net is eventually (frequently) in $\text{cl}(V)$ for each $V$ open about $x$.

Definition 2. A point $x$ in a topological space $X$ is in the $\theta$-closure [11, p. 103] of a set $K \subset X$ ($\theta$-cl($K$)) if $\text{cl}(V) \cap K \neq \emptyset$ for any $V$ open about $x$.

Definition 3. A subset $K$ of a topological space is $\theta$-closed [11, p. 103] if it contains its $\theta$-closure (i.e., $\theta$-cl($K$) $\subset K$).

The following theorems give some parallels of properties of closure and closed sets in a topological space for $\theta$-closure and $\theta$-closed sets in the space and some relationships between these notions. The proofs of these theorems are straightforward and are omitted [11, Lemmas 1, 2, 3].

Theorem 1. A point $x$ in a topological space is in the $\theta$-closure of a subset $K$ if and only if there is a net $x_{\alpha}$ in $K$ which $\theta$-converges to $x$ ($x_{\alpha} \theta \cdot x$).

Theorem 2. In any topological space
(a) the empty set and the whole space are $\theta$-closed,
(b) arbitrary intersections and finite unions of $\theta$-closed sets are $\theta$-closed,
(c) $\text{cl}(K) \subset \theta$-cl($K$) for each subset $K$,
(d) a $\theta$-closed subset is closed.

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Example 1. Each nonempty countable subset of the set of reals endowed with the co-countable topology is closed but not \( \theta \)-closed.

Main results. There are several characterizations of \( H \)-closed spaces in the literature [2, p. 145], [1, p. 97]. For a definition, we use the following:

Definition 4. A Hausdorff space \( X \) is \( H \)-closed if every open cover \( \mathcal{U} \) of \( X \) contains a finite subcollection \( \mathcal{B} \) such that \( \{ \text{cl}(V) : V \in \mathcal{B} \} \) covers \( X \).

We also make use of the following theorem immediately gotten from [11, Theorem 2]:

Theorem 3. A Hausdorff space is \( H \)-closed if and only if each net in the space has a \( \theta \)-convergent subnet.

Definition 5. A function \( g : X \to Y \) is weakly continuous [6, Theorem 6] if for each net \( x_n \) in \( X \) such that \( x_n \to x \), the net \( g(x_n) \to g(x) \).

Definition 6. A function \( g : X \to Y \) has a strongly-closed graph [6, p. 473] if for each \((x, y) \in (X \times Y) \setminus G(g)\), there are open sets \( U \) and \( V \) about \( x \) and \( y \), respectively, such that \((U \times \text{cl}(V)) \cap \text{G}(g) = \emptyset \).

It is known that a function \( g : X \to Y \) has a closed graph if and only if whenever a net \( x_n \to x \) in \( X \) and \( g(x_n) \to y \) in \( Y \), it follows that \( g(x) = y \) [13, p. 115]. We have the following similar result for functions with strongly-closed graphs.

Theorem 4. A function \( g : X \to Y \) has a strongly-closed graph if and only if whenever a net \( x_n \to x \) in \( X \) and \( g(x_n) \to y \) in \( Y \), it follows that \( g(x) = y \).

Proof. Let \( g \) have a strongly-closed graph and let \( x_n \) be a net in \( X \) satisfying \( x_n \to x \) and \( g(x_n) \to y \). Then \((V \times \text{cl}(W)) \cap \text{G}(g) \neq \emptyset \) for \( V, W \) open about \( x \) and \( y \) respectively. So, \((x, y) \in \text{G}(g) \) and \( g(x) = y \). For the converse, let \((x, y) \in (X \times Y) \setminus \text{G}(g) \). Then \( y \neq g(x) \), and there is no net \( x_n \) in \( X \) satisfying \( x_n \to x \) and \( g(x_n) \to y \). If \((V_\alpha \times \text{cl}(W_\beta)) \cap \text{G}(g) \neq \emptyset \) for each pair \( V_\alpha, W_\beta \) of sets open about \( x \) and \( y \) respectively, choose \((x_{\alpha, \xi}, g(x_{\alpha, \xi})) \in (V_\alpha \times \text{cl}(W_\beta)) \cap \text{G}(g) \). The ordering of \( \{ V_\alpha \times \text{cl}(W_\beta) : V_\alpha, W_\beta \text{ open about } x \text{ and } y \text{ respectively} \} \) by inclusion renders \((x_{\alpha, \xi}, g(x_{\alpha, \xi})) \) a net with \( x_{\alpha, \xi} \to x \) and \( g(x_{\alpha, \xi}) \to y \), a contradiction. Therefore, there are sets \( V, W \) open about \( x, y \), respectively, and satisfying \((V \times \text{cl}(W)) \cap \text{G}(g) = \emptyset \); and \( G(g) \) is strongly-closed. This completes the proof.

We may use the characterizations above to give a proof of the following theorem which is different and shorter than that given in [6]. If \((x_\alpha, D)\) is a net in a space \( X \), we will denote \( \{x_\alpha : \alpha > \mu \} \) by \( T_\mu \) for each \( \mu \in D \). Using this notation it is clear that \( x_\alpha \theta \)-converges (\( \theta \)-accumulates) to a point \( x \in X \) if for each open \( V \) about \( x \), there is a \( \mu \in D \) satisfying (each \( \mu \in D \) satisfies) \( T_\mu \subset \text{cl}(V) \) \((T_\mu \cap \text{cl}(V) \neq \emptyset)\). Let \( \mathcal{S} \) denote a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces.

Theorem 5. A Hausdorff space \( Y \) is \( H \)-closed if and only if for every space in class \( \mathcal{S} \), each \( g : X \to Y \) with a strongly-closed graph is weakly continuous.

Proof. Let \( Y \) be \( H \)-closed, let \( X \) be any space and let \( g : X \to Y \) have a strongly-closed graph. Let \( x_\alpha \to x \) in \( X \). Then \( g(x_\alpha) \) is a net in \( Y \), so there is a
subnet $y_\beta$ of $x_\alpha$ and $y \in Y$ with $g(y_\beta) \to y$. By Theorem 4, $g(x) = y$. Let $V$ be a regular open set about $g(x)$. If $g(x_\alpha)$ is not eventually in $\text{cl}(V)$, there is a subnet $z_\alpha$ of $x_\alpha$ such that $g(z_\alpha)$ $\theta$-converges to some $z \in Y - V$ since $Y - V$ is a regular closed set and thus $H$-closed. This then forces $g(x) \in Y - V$, a contradiction. So $g(x_\alpha) \to g(x)$. For the converse, let $x_0 \in Y$ and let $(x_\alpha, D)$ be a net in $Y - \{x_0\}$ with no $\theta$-accumulation point in $Y - \{x_0\}$. Let $X = \{x_\alpha: \alpha \in D\} \cup \{x_0\}$ with the topology generated by $\{\{x_\alpha\}: \alpha \in D\}$ and $\{T_\mu \cup \{x_0\}: \mu \in D\}$ as the basic open sets. $X$ is a Hausdorff door space \cite[p. 76]{7} and is easily shown to be in class $\mathcal{S}$. Let $i: X \to Y$ be the identity function and let $(x, y) \in (X \times Y) - G(i)$. If $x \neq x_0$, then $\{x\}$ is open in $X$; choose $V$ open in $Y$ about $y$ with $x \not\in \text{cl}(V)$. Then, clearly, $((\{x\} \times \text{cl}(V)) \cap G(i)) \cap \text{cl}(V) = \emptyset$. If $x = x_0$, then $y \neq x_0$; so there is a $\mu \in D$ and a $V$ open in $Y$ about $y$ satisfying $x_0 \not\in \text{cl}(V)$ and $T_\mu \cap \text{cl}(V) = \emptyset$. So $X - \text{cl}(V)$ is open in $X$ about $x$ and $(X - \text{cl}(V)) \cap \text{cl}(V) = \emptyset$. Thus, $i$ has a strongly-closed graph and is weakly continuous. Consequently, if $V$ is open about $x_0$, there is a $\mu \in D$ satisfying $T_\mu \subset \text{cl}(V)$ \cite[p. 44]{8}, so $x_\alpha \to x_0$. This completes the proof.

In \cite[p. 474]{6}, an example is given to show that the strongly-closed graph condition in Theorem 5 cannot be relaxed to a closed graph condition. This example was extracted from \cite{12} and is not described explicitly in \cite{6} presumably because of its somewhat complicated description. We now exhibit a space with a simpler description which meets the purposes of the example in \cite{6}.

**Example 2.** Let $N$ be the set of positive integers and let $X = \{0\} \cup [1, \infty)$ with the topology generated by the usual subspace topology of the reals on $[1, \infty)$ and $\{\{0\} \cup \bigcup_{m \in N} (k, k + 1): m \in N\}$ as basic open sets.

(a) The space $X$ is Hausdorff.

(b) The space $X$ is not compact since $N$ is an infinite subset of $X$ without accumulation points.

(c) The space $X$ is $H$-closed.

(d) The function $g$, from $\{1 + 1/n: n \in N\} \cup \{1\}$ with the subspace topology, defined by $g(1) = 1$ and $g(1 + 1/n) = n$ for each $n \in N$ has a closed graph which is not strongly-closed. Also, $g$ is not weakly continuous at 1.

In \cite{3}, \cite{4}, \cite{5}, and \cite{10}, theorems of the following form are proved: $X$ has property $\lambda$ if and only if the projection $\pi_y: X \times Y \to Y$ is closed for each space $Y$ in a certain class. The next four theorems and main results give an analogue of this form for $H$-closed spaces.

**Theorem 6.** If $X$ is an $H$-closed space then the projection from $X \times Y$ onto $Y$ takes $\theta$-closed subsets onto $\theta$-closed subsets for any space $Y$.

**Proof.** Let $X$ be $H$-closed, let $Y$ be any space and let $K \subset X \times Y$ be $\theta$-closed. Let $y \in \theta - \text{cl}(\pi_y(K))$. There is a net $(x_\alpha, y_\alpha) \in K$ with $y_\alpha \to y$. There is a subnet $x_{\alpha'}$ of $x_\alpha$ and $x \in X$ with $x_{\alpha'} \to x$. So $(x_{\alpha'}, y_{\alpha'}) \to (x, y)$ and $(x, y) \in \theta - \text{cl}(K) \subset K$. Thus $y \in \pi_y(K)$.

**Theorem 7.** If $X$ is a Hausdorff space and the projection from $X \times Y$ onto $Y$
takes \( \theta \)-closed subsets onto closed subsets for every space \( Y \), then \( X \) is \( H \)-closed.

**Proof.** Let \((y_\alpha, D)\) be a net in \( X \) with no \( \theta \)-convergent subnet and let \( y_0 \notin X \). Let \( Y = \{ y_\alpha : \alpha \in D \} \cup \{ y_0 \} \) with the topology generated by \( \{ \{ y_\alpha \} : \alpha \in D \} \) and \( \{ T_\mu \cup \{ y_0 \} : \mu \in D \} \) as basic open sets. Let \( K = \{ (y_\alpha, y_\alpha) : \alpha \in D \} \) and let \((a, b) \in (X \times Y) - K \). Then \( a \neq y_0 \) and \( a \neq b \). Let \( V \) be open about \( a \) satisfying \( \{ b, y_0 \} \subset Y - \text{cl}(V) \) and \( T_\mu \subset Y - \text{cl}(V) \) for some \( \mu \in D \). Then \( Y - \text{cl}(V) \) is open and closed in \( Y \) and so \( X \times (Y - \text{cl}(V)) \) is open about \((a, b)\). Also,

\[
\text{cl} [X \times (Y - \text{cl}(V))] \cap K = (\text{cl}(V) \times (Y - \text{cl}(V))) \cap K = \emptyset.
\]

Thus, \((a, b) \in \theta \text{-cl}(K)\) and \( K \) is \( \theta \)-closed. \( \pi_\gamma(K) \) is therefore closed in \( Y \) and \( y_0 \in \text{cl}(\pi_\gamma(K)) \). This is a contradiction establishing the result.

Combining Theorems 6 and 7, we get the promised result.

**Theorem 8.** A Hausdorff space \( X \) is \( H \)-closed if and only if for every space \( Y \), the projection from \( X \times Y \) onto \( Y \) takes \( \theta \)-closed subsets onto \( \theta \)-closed subsets.

Noting that the space \( Y \) used in the proof of Theorem 7 is a Hausdorff door space and is in the class \( S \) whose description precedes Theorem 5, we may state the following theorem.

**Theorem 9.** A Hausdorff space \( X \) is \( H \)-closed if and only if for every Hausdorff door space (space in class \( S \)) \( Y \), the projection from \( X \times Y \) onto \( Y \) takes \( \theta \)-closed subsets onto \( \theta \)-closed subsets.

**References**


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