A FIXED POINT THEOREM FOR A SYSTEM OF MULTIVALUED TRANSFORMATIONS

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Abstract. We shall prove a fixed point theorem for a system of multivalued mappings which generalizes the result obtained by the author [1, Theorem 1]. For \( n = 1 \) we obtain a generalization of results of Reich [5, Theorem 5] and Nadler [3, Theorem 5], [4, Theorem 1].

1. Let \((X, d)\) be a metric space. We follow the notation of [4].
   (a) \( CL(X) = \{ C: C \text{ is a nonempty closed subset of } X \} \),
   (b) \( N(\epsilon, C) = \{ x \in X: d(x, c) < \epsilon \text{ for some } c \in C \} \), \( \epsilon > 0 \), \( C \in CL(X) \),
   \( H(A, B) = \inf \{ \epsilon > 0: A \subset N(\epsilon, B) \text{ and } B \subset N(\epsilon, A) \} \), if the infimum exists,
   \( \infty \), otherwise,
   \( A, B \in CL(X) \).

   The function \( H \) is called the generalized Hausdorff distance for \( CL(X) \) induced by \( d \). \( D(x, A) \) will denote the ordinary distance between \( x \in X \) and \( A \in CL(X) \).

2. We follow the notation of [2].

\[
\begin{align*}
\epsilon_{i,k}^1 &= \begin{cases} 
c_{i,k} & \text{for } i \neq k, \\
n_{i,k} & \text{for } i = k,
\end{cases} \\
\epsilon_{i,k}^{s+1} &= \begin{cases} 
c_{i,1}^s c_{i+1,k+1}^s + c_{i+1,1}^s c_{i,k+1}^s & \text{for } i \neq k, \\
c_{i,1}^s c_{i+1,k+1}^s - c_{i+1,1}^s c_{i,k+1}^s & \text{for } i = k,
\end{cases}
\end{align*}
\]

(2) \( s = 1, \ldots, n - 1, i, k = 1, \ldots, n - s \).

The following result is contained in [2].

Lemma. Let \( \epsilon_{i,k}^1 > 0 \), \( i, k = 1, \ldots, n \). The system of inequalities

\[
\sum_{k=1}^{n} \epsilon_{i,k}^1 r_k < r_i, \quad i = 1, \ldots, n,
\]

has a solution \( r_i > 0 \), \( i = 1, \ldots, n \), if and only if the following inequalities hold:

Received by the editors January 28, 1975.
Key words and phrases. Multivalued mappings, iteration, fixed point theorems.
(4) \[ c_{i,s}^s > 0, \quad s = 1, \ldots, n, \quad i = 1, \ldots, n + 1 - s. \]

Suppose that \( r_i > 0, \quad i = 1, \ldots, n, \) is the solution of the system of inequalities (3). We define

(5) \[ v = \max_i \left( r_i^{-1} \sum_{k=1}^n c_{i,k} r_k \right). \]

In view of the homogeneity of the system of inequalities (3), definition (5) is correct and

(6) \[ 0 < v < 1. \]

Let \( c \) be a real number such that

(7) \[ 0 < c < 1 - v. \]

Let \( (X_i, d_i), \quad i = 1, \ldots, n, \) be metric spaces. \( H_i(A, B), \quad i = 1, \ldots, n, \) will denote the Hausdorff distance between two elements of \( CL(X_i), \quad i = 1, \ldots, n, \) obtained from \( d_i, \quad i = 1, \ldots, n, \) and \( D_j(x, A) \) will denote the ordinary distance between \( x \in X_j, \quad A \in CL(X_j), \quad i = 1, \ldots, n. \)

Now we shall prove the following

**Theorem.** Let \( (X_i, d_i), \quad i = 1, \ldots, n, \) be complete metric spaces and let \( a_{i,k} > 0, \quad b_{i,k} > 0 \) for \( i, \quad k = 1, \ldots, n. \) Let \( c_{i,k} = a_{i,k} + b_{i,k}, \quad i, \quad k = 1, \ldots, n, \) be positive and let the numbers \( c_{i,s}^s, \quad s = 1, \ldots, n, \quad i, \quad k = 1, \ldots, n + 1 - s, \) defined by (1) and (2) fulfil the inequalities (4). Suppose that the transformations \( F_i: X_1 \times \cdots \times X_n \to CL(X_i), \quad i = 1, \ldots, n, \) fulfil

(8) \[ H_i[F_i(x_1, \ldots, x_n), F_i(z_1, \ldots, z_n)] \leq \sum_{k=1}^n a_{i,k} d_k(x_k, z_k) \]

\[ + \sum_{k=1}^n b_{i,k} D_k[x_k, F_k(x_1, \ldots, x_n)] \]

\[ + cD_i[z_i, F_i(z_1, \ldots, z_n)] \]

for all \( x_j, z_j \in X_j, \quad i,j = 1, \ldots, n, \) where \( c \) fulfils (7). Then the system \( (F_1, \ldots, F_n) \) has a fixed point, i.e. there exist points \( u_i \in X_i, \quad i = 1, \ldots, n, \) such that \( u_i \in F_i(u_1, \ldots, u_n) \) for all \( i = 1, \ldots, n. \)

**Proof.** Let \( x_0^i \in X_i, \quad i = 1, \ldots, n, \) and choose \( x_i^1 \in F_i(x_0^1, \ldots, x_0^n), \quad i = 1, \ldots, n. \) From (1), (2), (4), the Lemma and (5) we may choose a system of positive numbers \( r_1, \ldots, r_n \) such that

(9) \[ \sum_{k=1}^n c_{i,k} r_k \leq v r_i, \quad i = 1, \ldots, n. \]

We may assume (from the homogeneity of the above system) that

(10) \[ d_i(x_0^i, x_i^1) < r_i \quad \text{and} \quad r_i > 1 \quad \text{for} \quad i = 1, \ldots, n. \]

Let \( A, B \in CL(X_i) \) and let \( a \in A. \) By definition, if \( q > 0, \) then there exists \( b \in B \) such that \( d_i(a, b) < H_i(A, B) + q. \) Hence in view of conditions \( F_i(x_0^1, \ldots, x_0^n), \quad F_i(x_1^1, \ldots, x_1^n) \in CL(X_i) \) and \( x_i^1 \in F_i(x_0^1, \ldots, x_0^n), \quad i = 1, \ldots, n, \) there exist points \( x_i^2 \in F_i(x_1^1, \ldots, x_1^n), \quad i = 1, \ldots, n, \) such that
By induction, we obtain the sequences \( \{x_i^k\}_{k=1}^{\infty}, i = 1, \ldots, n, \) of points of \( X_i, \) such that \( x_i^k \in F_i(x_i^{k-1}, \ldots, x_i^{k-1}), i = 1, \ldots, n, k = 1, 2, \ldots, \) and

\[
d_i(x_i^k, x_i^{k+1}) \leq H_i\left[ F_i(x_i^{k-1}, \ldots, x_i^{k-1}), F_i(x_i^1, \ldots, x_i^n) \right] + \frac{\nu^k}{(1 - c)^{k-1}}, \quad i = 1, \ldots, n, \quad k = 1, 2, \ldots.
\]

From (11), (8), (10) and (9) we obtain

\[
d_i(x_i^1, x_i^2) \leq \sum_{k=1}^{n} a_{i,k} d_k(x_i^0, x_i^1) + \sum_{k=1}^{n} b_{i,k} D_k\left[ x_i^0, F_k(x_i^0, \ldots, x_i^0) \right] + cD_i\left[ x_i^1, F_i(x_i^1, \ldots, x_i^1) \right] + \nu
\]

\[
\leq \sum_{k=1}^{n} (a_{i,k} + b_{i,k}) d_k(x_i^0, x_i^1) + c d_i(x_i^1, x_i^2) + \nu
\]

\[
\leq \sum_{k=1}^{n} c_i r_k + c d_i(x_i^1, x_i^2) + \nu
\]

Thus

\[
d_i(x_i^1, x_i^2) \leq \frac{\nu}{1 - c} r_i + \frac{\nu}{1 - c} \leq 2 \frac{\nu}{1 - c} r_i.
\]

Recalling (12), (8), (10), (9) and the induction principle, we obtain

\[
d_i(x_i^k, x_i^{k+1}) \leq (k + 1)(\nu/(1 - c))^k r_i, \quad i = 1, \ldots, n, \quad k = 1, 2, \ldots.
\]

Now we have

\[
d_i(x_i^1, x_i^{k+m}) \leq d_i(x_i^1, x_i^{k+1}) + \cdots + d_i(x_i^{k+m-1}, x_i^{k+m})
\]

\[
\leq (k + 1)\left(\frac{\nu}{1 - c}\right)^{k-1} r_i + \cdots + (k + m)\left(\frac{\nu}{1 - c}\right)^{k+m-1} r_i,
\]

\[
\leq \sum_{s=k}^{\infty} (s + 1)\alpha^s r_i \leq (k + 2)\alpha^k (1 - \alpha)^{-2} r_i
\]

for all \( k,m \geq 1, i = 1, \ldots, n, \) where \( \alpha = \nu/(1 - c). \) Thus \( \{x_i^k\}_{k=1}^{\infty}, i = 1, \ldots, n, \) are Cauchy sequences and, therefore, \( x_i^k \to u_i \in X_i, i = 1, \ldots, n. \) We claim that \((u_1, \ldots, u_n)\) is a fixed point of \((F_1, \ldots, F_n).\)

Actually

\[
D_i\left[ u_i, F_i(u_1, \ldots, u_n) \right] \leq d_i(u_i, x_i^{k+1}) + D_i\left[ x_i^{k+1}, F_i(u_1, \ldots, u_n) \right]
\]

\[
\leq d_i(u_i, x_i^{k+1}) + H_i\left[ F_i(x_i^1, \ldots, x_i^n), F_i(u_1, \ldots, u_n) \right]
\]

\[
\leq d_i(u_i, x_i^{k+1}) + \sum_{s=1}^{n} a_{i,s} d_s(x_s^k, u_s)
\]
\[
+ \sum_{s=1}^{n} b_{i,s} D_s \left[ x_s^k, F_s \left( x_s^1, \ldots, x_s^n \right) \right] + c D_i \left[ u_i, F_i (u_1, \ldots, u_n) \right] \\
\leq d_i (u_i, x_i^{k+1}) + \sum_{s=1}^{n} a_{i,s} d_s \left( x_s^k, u_s \right) \\
+ \sum_{s=1}^{n} b_{i,s} d_s \left( x_s^k, x_s^{k+1} \right) + c D_i \left[ u_i, F_i (u_1, \ldots, u_n) \right].
\]

Hence,
\[
D_i \left[ u_i, F_i (u_1, \ldots, u_n) \right] \\
\leq \frac{1}{1 - c} \left[ d_i (u_i, x_i^{k+1}) + \sum_{s=1}^{n} a_{i,s} d_s \left( x_s^k, u_s \right) \\
+ \sum_{s=1}^{n} b_{i,s} d_s \left( x_s^k, x_s^{k+1} \right) \right] \to 0 \quad \text{as } k \to \infty.
\]

Since \( F_i (u_1, \ldots, u_n) \) is closed, this means that \( u_i \in F_i (u_1, \ldots, u_n), \) \( i = 1, \ldots, n, \) which completes the proof.

**References**


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