A FIXED POINT THEOREM FOR A SYSTEM OF MULTIVALUED TRANSFORMATIONS

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Abstract. We shall prove a fixed point theorem for a system of multivalued mappings which generalizes the result obtained by the author [1, Theorem 1]. For \( n = 1 \) we obtain a generalization of results of Reich [5, Theorem 5] and Nadler [3, Theorem 5], [4, Theorem 1].

1. Let \((X, d)\) be a metric space. We follow the notation of [4].

(a) \(CL(X) = \{ C: C \) is a nonempty closed subset of \( X \}\),

(b) \(N(\varepsilon, C) = \{ x \in X: d(x, c) < \varepsilon \) for some \( c \in C \}, \varepsilon > 0, C \in CL(X), \)

\[ H(A, B) = \left\{ \begin{array}{ll}
\inf\{\varepsilon > 0: A \subseteq N(\varepsilon, B) \) and \( B \subseteq N(\varepsilon, A)\}, & \text{if the infimum exists,} \\
\infty, & \text{otherwise,}
\end{array} \right. \]

\( A, B \in CL(X). \)

The function \( H \) is called the generalized Hausdorff distance for \( CL(X) \) induced by \( d \). \( D(x, A) \) will denote the ordinary distance between \( x \in X \) and \( A \in CL(X). \)

2. We follow the notation of [2].

\[ c_{i,k} = \left\{ \begin{array}{ll}
c_{i,k}, & \text{for } i \not= k, \\
1 - c_{i,k}, & \text{for } i = k,
\end{array} \right. \quad i, k = 1, \ldots, n,
\]

\[ c_{i,k}^{s+1} = \left\{ \begin{array}{ll}
c_{1,1}c_{i,k+1}^{s} + c_{1,1}c_{i,k+1}^{s} & \text{for } i \not= k, \\
c_{1,1}c_{i,k+1}^{s} + c_{1,1}c_{i,k+1}^{s} - c_{i+1,1}c_{i+1,k+1}^{s} & \text{for } i = k,
\end{array} \right. \]

\[ s = 1, \ldots, n - 1, i, k = 1, \ldots, n - s. \]

The following result is contained in [2].

Lemma. Let \( c_{i,k}^{1} > 0, i, k = 1, \ldots, n \). The system of inequalities

\[ \sum_{k=1}^{n} c_{i,k} r_{k} < r_{i}, \quad i = 1, \ldots, n, \]

has a solution \( r_{i} > 0, i = 1, \ldots, n \), if and only if the following inequalities hold:
(4) \( c_{i,k}^s > 0, \quad s = 1, \ldots, n, \quad i = 1, \ldots, n + 1 - s. \)

Suppose that \( r_i > 0, \ i = 1, \ldots, n, \) is the solution of the system of inequalities (3). We define

\[
\nu = \max_i \left( r_i^{-1} \sum_{k=1}^n c_{i,k} r_k \right).
\]

In view of the homogeneity of the system of inequalities (3), definition (5) is correct and

\[
0 < \nu < 1.
\]

Let \( c \) be a real number such that

\[
0 < c < 1 - \nu.
\]

Let \((X_i, d_i), \ i = 1, \ldots, n,\) be metric spaces. \( H_i(A, B), \ i = 1, \ldots, n,\) will denote the Hausdorff distance between two elements of \( CL(X_i), \ i = 1, \ldots, n,\) obtained from \( d_i, \ i = 1, \ldots, n,\) and \( D_j(x, A)\) will denote the ordinary distance between \( x \in X_j, \ A \in CL(X_j), \ i = 1, \ldots, n.\)

Now we shall prove the following

**Theorem.** Let \((X_i, d_i), \ i = 1, \ldots, n,\) be complete metric spaces and let \( a_{i,k} > 0, b_{i,k} > 0 \) for \( i, k = 1, \ldots, n.\) Let \( c_{i,k} = a_{i,k} + b_{i,k}, i, k = 1, \ldots, n,\) be positive and let the numbers \( c_{i,k}^s, \ s = 1, \ldots, n, \ i, k = 1, \ldots, n + 1 - s,\) defined by (1) and (2) fulfil the inequalities (4). Suppose that the transformations \( F_i : X_1 \times \cdots \times X_n \to CL(X_i), \ i = 1, \ldots, n,\) fulfil

\[
H_j([F_j(x_1, \ldots, x_n), F_j(z_1, \ldots, z_n)]) \leq \sum_{k=1}^n a_{i,k} d_k(x_k, z_k) + \sum_{k=1}^n b_{i,k} D_k(x_k, F_k(x_1, \ldots, x_n)) + cD_l(z_l, F_l(z_1, \ldots, z_n))
\]

for all \( x_j, z_j \in X_j, \ i, j = 1, \ldots, n,\) where \( c \) fulfils (7). Then the system \((F_1, \ldots, F_n)\) has a fixed point, i.e. there exist points \( u_i \in X_i, \ i = 1, \ldots, n,\) such that \( u_i \in F_i(u_1, \ldots, u_n) \) for all \( i = 1, \ldots, n.\)

**Proof.** Let \( x_0^i \in X_i, \ i = 1, \ldots, n,\) and choose \( x_i^1 \in F_i(x_0^1, \ldots, x_0^n), i = 1, \ldots, n.\) From (1), (2), (4), the Lemma and (5) we may choose a system of positive numbers \( r_1, \ldots, r_n\) such that

\[
\sum_{k=1}^n c_{i,k} r_k \leq \nu r_i, \quad i = 1, \ldots, n.
\]

We may assume (from the homogeneity of the above system) that

\[
d_i(x_0^i, x_i^1) < r_i \quad \text{and} \quad r_i > 1 \quad \text{for} \quad i = 1, \ldots, n.
\]

Let \( A, B \in CL(X_i)\) and let \( a \in A.\) By definition, if \( q > 0,\) then there exists \( b \in B\) such that \( d_i(a, b) < H_i(A, B) + q.\) Hence in view of conditions \( F_i(x_1^0, \ldots, x_n^0), \ F_i(x_1^1, \ldots, x_n^1) \in CL(X_i)\) and \( x_i^1 \in F_i(x_1^0, \ldots, x_n^0), i = 1, \ldots, n,\) there exist points \( x_i^2 \in F_i(x_1^1, \ldots, x_n^1), i = 1, \ldots, n,\) such that
By induction, we obtain the sequences \( \{x^i_k\}_{k=1}^\infty, i = 1, \ldots, n, \) of points of \( X_i \), \( i = 1, \ldots, n, \) such that \( x^i_k \in F_i(x^i_{k-1}, \ldots, x^i_{k-1}), i = 1, \ldots, n, \) and

\[
d_i(x^i_k, x^i_{k+1}) \leq H_i[F_i(x^i_{k-1}, \ldots, x^i_{k-1}), F_i(x^i_1, \ldots, x^i_n)] + \nu.
\]

(11)

From (11), (8), (10) and (9) we obtain

\[
d_i(x^1_t, x^2_t) \leq \sum_{k=1}^n a_{i,k}d_k(x^0_k, x^1_k) + \sum_{k=1}^n b_{i,k}D_k[x^0_k, F_k(x^1_1, \ldots, x^0_n)] + cD_t[x^1_1, F_t(x^1_1, \ldots, x^1_t)] + \nu
\]

\[
\leq \sum_{k=1}^n (a_{i,k} + b_{i,k})d_k(x^0_k, x^1_k) + c d_t(x^1_1, x^1_t) + \nu
\]

\[
\leq \sum_{k=1}^n c_{i,k}r_k + c d_t(x^1_1, x^1_t) + \nu
\]

\[
\leq \nu r_t + c d_t(x^1_1, x^1_t) + \nu.
\]

Thus

\[
d_i(x^1_t, x^2_t) \leq \frac{\nu}{1-c} r_t + \frac{\nu}{1-c} \leq 2 \frac{\nu}{1-c} r_t.
\]

Recalling (12), (8), (10), (9) and the induction principle, we obtain

\[
d_i(x^i_k, x^i_{k+1}) \leq (k+1)(\nu/(1-c))^k r_t, \quad i = 1, \ldots, n, \quad k = 1, 2, \ldots.
\]

Now we have

\[
d_i(x^i_k, x^i_{k+m}) \leq d_i(x^i_k, x^i_{k+1}) + \cdots + d_i(x^i_{k+m-1}, x^i_{k+m})
\]

\[
\leq (k+1)\left(\frac{\nu}{1-c}\right)^k r_i + \cdots + (k+m)\left(\frac{\nu}{1-c}\right)^{k+m-1} r_i
\]

\[
\leq \sum_{s=k}^{\infty} (s+1) \alpha^s r_i \leq (k+2)\alpha^k (1-\alpha)^{-2} r_i
\]

for all \( k, m \geq 1, \quad i = 1, \ldots, n, \) where \( \alpha = \nu/(1-c) \). Thus \( \{x^i_k\}_{k=1}^\infty, i = 1, \ldots, n, \) are Cauchy sequences and, therefore, \( x^i_k \to u_i \in X_i, \quad i = 1, \ldots, n. \) We claim that \( (u_1, \ldots, u_n) \) is a fixed point of \( (F_1, \ldots, F_n). \)

Actually

\[
D_i[u_i, F_i(u_1, \ldots, u_n)] \leq d_i(u_i, x^i_{k+1}) + D_i[x^i_{k+1}, F_i(u_1, \ldots, u_n)]
\]

\[
\leq d_i(u_i, x^i_{k+1}) + H_i[F_i(x^i_k, \ldots, x^i_n), F_i(u_1, \ldots, u_n)]
\]

\[
\leq d_i(u_i, x^i_{k+1}) + \sum_{s=1}^n a_{i,s}d_s(x^s_k, u_s)
\]
\[ + \sum_{s=1}^{n} b_{i,s} D_s \left[ x_s^k, F_s \left( x_1^k, \ldots, x_n^k \right) \right] + cD_i \left[ u_i, F_i \left( u_1, \ldots, u_n \right) \right] \]

\[ \leq d_i \left( u_i, x_i^{k+1} \right) + \sum_{s=1}^{n} a_{i,s} d_s \left( x_s^k, u_s \right) \]

\[ + \sum_{s=1}^{n} b_{i,s} d_s \left( x_s^k, x_s^{k+1} \right) + cD_i \left[ u_i, F_i \left( u_1, \ldots, u_n \right) \right]. \]

Hence,

\[ D_i \left[ u_i, F_i \left( u_1, \ldots, u_n \right) \right] \]

\[ \leq \frac{1}{1 - c} \left[ d_i \left( u_i, x_i^{k+1} \right) + \sum_{s=1}^{n} a_{i,s} d_s \left( x_s^k, u_s \right) \right. \]

\[ \left. + \sum_{s=1}^{n} b_{i,s} d_s \left( x_s^k, x_s^{k+1} \right) \right] \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]

Since \( F_i \left( u_1, \ldots, u_n \right) \) is closed, this means that \( u_i \in F_i \left( u_1, \ldots, u_n \right), \quad i = 1, \ldots, n \), which completes the proof.

**References**


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