THE GENERAL SOLUTION OF A FIRST ORDER DIFFERENTIAL POLYNOMIAL

RICHARD M. COHN

ABSTRACT. A purely algebraic proof is given of a theorem, proved analytically by Ritt, which determines the number of derivations needed to find a basis for the perfect ideal of the general solution of an algebraically irreducible first order differential polynomial.

1. It was shown by J. F. Ritt [2, 129] that to separate the singular components from the general solution of an irreducible first order differential polynomial $G$ of degree $m$ in the first derivative of the indeterminate it is sufficient to decompose the system formed by $G$ and its first $m - 1$ derivatives treated as algebraic polynomials. His proof uses analytic methods. Kolchin [1, p. xiv] observed that this result is valid in abstract differential algebra because of the "differential Lefschetz principle" due to Seidenberg [3, p. 160, embedding theorem], but that no purely algebraic proof is known. In this note I give such a proof of Ritt's theorem. The mechanics of the argument are the same as in Ritt's work, except that I have preferred to introduce a parameter to permit power series with integral rather than fractional exponents.

2. Let $K$ be an ordinary differential field of characteristic 0 with derivation $D$, and let $Y$ be a differential indeterminate over $K$. Derivatives in $K\{Y\}$ will frequently be denoted by subscripts.

Theorem. Let $G \in K\{Y\}$ be of order 1 and degree $m$ in $Y_1$. Let $G$ have no factor of order 0 and no factor in common with $\partial G/\partial Y_1$. Let $P_1, \ldots, P_k$ be those minimal prime divisors of the ideal $(G, \ldots, G_{m-1})$ of $K[Y, \ldots, Y_m]$ which do not contain $\partial G/\partial Y_1$. Then no solution in a component of order 0 of the manifold of $G$ as a differential polynomial annuls any $P_i$.

Remark. If $G$ is irreducible, then $k = 1$ and $P_1$ is the intersection with $K[Y, \ldots, Y_m]$ of the differential ideal of the general solution of $G$ [2, p. 130]. The components of order 0 are, of course, precisely the singular components. It follows readily that in the general case $k$ is the number of irreducible factors of $G$, and each $P_i$ is the intersection with $K[Y, \ldots, Y_m]$ of the differential ideal of the general solution of an irreducible factor. It follows

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further that no $P_s$ contains a polynomial of order 0.

**Proof.** Let $a$ be a solution in a zero order component of the manifold of $G$. We shall assume that $a$ annuls some $P_s$, say $P_1$, and obtain a contradiction. Note that without loss of generality we may assume $a = 0$. Indeed, if we enlarge $K$ to a differential field $K_1$ the hypotheses concerning factorization of $G$ remain valid in $K_1\{Y\}$. From standard results concerning the effect of ground field extensions we see that $P_1$ splits into prime components in $K_1[Y, \ldots, Y_m]$ no one of which contains $\partial G/\partial Y_1$. The analogous theorem of differential algebra [1, Chapter III, Proposition 3] shows that $a$ is in a zero order component of $G$ as a polynomial in $K_1\{Y\}$. Choosing $K_1$ to contain $a$ and then making the substitution $Y \rightarrow Y + a$ accomplishes the desired reduction. Henceforth, let $a = 0$.

We introduce a parameter $t$ and form the power series ring $K[[t]]$ and its quotient field $K((t))$. These are not differential. However, for $f \in K((t))$ define $\delta f$ to be the result of applying the derivation $D$ of $K$ to every coefficient of $f$, and $df/dt$ to be the formal derivative of $f$ with respect to $t$. Let $h \in K((t))$. Define $D_h f = \delta f + h\, df/dt$. Then $D_h$ is a derivation of $K((t))$.

Since $P_1$ contains no polynomial of order 0, $Y\partial G/\partial Y_1 \not\in P_1$. By hypothesis $P_1$ admits the solution $Y_i = 0, 0 \leq i \leq m$. It follows that $P_1$ has a solution $Y_i = f_i, 0 < i < m$, not annulling $Y\partial G/\partial Y_1$, where the $f_i$ are in $K((t))$ and begin with terms of positive degree. Concerning this solution we make the following observations.

(a) The $f_i$ annul $G_1, \ldots, G_{m-1}$. This requirement uniquely determines $f_2, \ldots, f_m$ when $f_0$ and $f_1$ are given. This is so since for $1 \leq i \leq m - 1$, $G_i = Y_{i+1}\partial G/\partial Y_1 +$ terms free of $Y_j, j > i$, and since $f_0, f_1$ do not annul $\partial G/\partial Y_1$.

(b) Choose $h$ so that $D_h f_0 = f_1$. (This is possible since $df_0/dt \neq 0$.) Regarding $K((t))$ as a differential field with the derivation $D_h$ we see that $f_0$ is a solution of $G$ as a differential polynomial. Hence, $Y_i = D_h f_0, 0 \leq i < m$, is a solution of the $G_i, 0 < i < m$.

Combining (a) and (b) we have

\[(A) \quad f_i = D_h^{i} f_0, \quad 0 < i < m.\]

Let $d$ denote the degree of the initial term of $h$. We shall show that $d < 0$. Let $I$ be the prime differential ideal of $K\{Y\}$ with generic zero $f_0$. (The derivation is, of course, $D_h$.) Since $f_0$ actually involves $t$, $I$ is not of order 0. Since $G \in I$, this shows that $I$ is of order 1 and is the ideal of a component of the manifold of $G$. If $d > 0$, then it is easy to see that each $D_h^i f_0, i = 0, 1, 2, \ldots$, begins with a term of positive degree in $t$. This implies that 0 is a solution of $I$, contradicting the assumption that 0 lies in a component of order 0 of the manifold of $G$.

**Remark.** One could also show $d < 0$ (that is, that $f_i$ begins with a term of lower degree than $f_0$) from the fact that $G$ must satisfy the low power condition with respect to $Y$. Conversely, noting that the result $d < 0$ applies to an arbitrary solution of $G$ (as a polynomial in $Y$ and $Y_1$) in series of positive powers of $t$, and examining the Newton polygon of $G$, one can prove the necessity of the low power condition for differential polynomials of first order.
From $d \leq 0$ it follows easily that if $f \in K((t))$ has an initial term of degree $a > 0$, then $D_h f$ has an initial term of degree $a + d - 1 < a$. Let $f_0, f_1, f_m$ begin with terms of degrees $a, b, c$ respectively. Using (A) and the preceding observation we find $a - b = 1 - d, a - c = m(1 - d)$. That is, $c = a - m(a - b)$.

$G$ must contain two terms with distinct power products $Y_1^p Y_0^q$ and $Y_1^r Y_1^s$ which yield terms of the same degree after the substitutions $Y = t^a, Y_1 = t^b$. Otherwise $f_0, f_1$ could not annul $G$. Therefore, $pa + qb = ra + sb$. Let, say, $q > s$. ($q = s$ is not possible, since it implies $r = p$.) Putting $q - s = n$ we rewrite the preceding equation as $nb = (r - p)a$. Then $r - p < n$, since $b < a$. Hence, $nb < (n - 1)a$. By hypothesis $q \leq m$, and so $n \leq m$. Then $mb < (m - 1)a$. Together with the result of the preceding paragraph this implies $c < 0$. This is a contradiction, and the proof is complete.

**Corollary.** Let $G$ be an algebraically irreducible differential polynomial of $K(Y)$ which is of order 1 and degree $m$ in $Y_1$. Let $I$ be the differential ideal of the general component of the manifold of $G$. Let $P$ be the unique minimal prime divisor of $(G, \ldots, G_{m-1})$ which does not contain $3G/3Y_1$. Then $I = \{ P \}$, and $I$ has a basis as a perfect differential ideal consisting of polynomials of order not exceeding $m$.

**Proof.** By [2, p. 30], $I \supset \{ P \}$; and, of course, $\{ P \} \supset \{ G \}$. It follows from the Theorem that no minimal prime divisor of $\{ G \}$ except $I$ contains $P$. Hence $I = \{ P \}$. A basis for $P$ as a polynomial ideal is a basis for $\{ P \}$ as a perfect differential ideal.

**References**


Department of Mathematics, Rutgers University, New Brunswick, New Jersey 08903