ARITHMETIC MEANS OF FOURIER COEFFICIENTS

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Abstract. Given the Fourier coefficients of an even continuous function, we find a necessary and sufficient condition such that their arithmetic means are equivalent to functions of odd continuous function. A similar result is shown for those Lipschitz classes whose elements are automatically equivalent to continuous functions.

Introduction. Let $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} b_n \sin nx$ be the Fourier series of $f_c$ and $f_s$, respectively. Hardy showed that $\sum_{n=1}^{\infty} [(a_1 + \cdots + a_n)/n] \cos nx$ and $\sum_{n=1}^{\infty} [(b_1 + \cdots + b_n)/n] \sin nx$ [2, p. 51] will also be Fourier series—let $T_{Hf_c}$ and $T_{Hf_s}$, respectively, represent these two series. It was shown by M. Kinukawa and S. Igar [5, Theorem 2] that $f_c \in L^\infty \Rightarrow T_{Hf_c} \in L^\infty$. In this paper we show that if $f_c \in C$, then $f_c(0) = 0 \Leftrightarrow T_{Hf_c} \in C$.

A. Konyushkov [6, Theorem 15] showed that for $0 < \alpha < 1/p < 1, f_c \in \Lambda_\alpha^p \Rightarrow T_{Hf_c} \in \Lambda_\alpha^p$. In this paper we consider the case $0 < 1/p < \alpha < 1$, and show that if $f_c \in \Lambda_\alpha^p$, then $\sum_{n=1}^{\infty} a_n = 0 \Leftrightarrow T_{Hf_c} \in \Lambda_\alpha^p$.

All the notations, unless otherwise mentioned, are taken from [9]. Every function is supposed to be defined a.e., periodic with period $2\pi$ and integrable on $[-\pi, \pi]$. We shall not distinguish between equivalent functions. Given a function $f$, by $f(t)$ we would mean $\lim_{n \to \infty} \sigma_n(t; f)$ whenever it exists. For convenience we shall write the Lipschitz classes $\Lambda_\alpha^p$ and $\Lambda_\alpha$ as $\Lambda(\alpha, p)$ and $\Lambda(\alpha)$, respectively.

Definition. For $f$ and $\tilde{f} \in L^1$, we define $T_{Sf}$ as the function representing the Fourier series $\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sigma_n(0; f)/n] \sin nt$.

We can easily check the following:

(i) $T_{Sf_c} = T_{Hf_c}$.
(ii) For $0 < \alpha < 1 < p < \infty$ and $\alpha p \neq 1$,$\Rightarrow T_{Sf_c} \in \Lambda(\alpha, p) \Leftrightarrow T_{Hf_c} \in \Lambda(\alpha, p)$.
(iii) For $0 < 1/p < \alpha < 1$, by [4, Theorem 5 (ii)],

$$f_c \in \Lambda(\alpha, p) \Rightarrow \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sigma_n(0; f) = f(0).$$

Now we state our results in the following form.

Theorem 1. Let $f \in C$. Then $f(0) = 0 \Leftrightarrow T_{Sf} \in C$.
Theorem 2. Let $f \in \Lambda(\alpha, p)$ where $0 < 1/p < \alpha < 1$. Then $f(0) = 0 \iff T_{sf} \in \Lambda(\alpha, p)$.

Lemma 1.1. For $1 < p \leq \infty$, $\|T_{sf}\|_p \leq A_p \|f\|_p$ where $A_p$ is a constant depending upon $p$ only.

Proof. Let $f \sim \sum_{n=0}^\infty (a_n \cos nt + b_n \sin nt)$. Take $f_c$ and $f_s$ as before. Then

$$T_{sf}(t) = T_{Ht}c(t) + a_0(\text{sgn } t)(\pi - |t|)/2 \text{ for a.e. } t.$$  

Hence, for $1 < p < \infty$, by [2, p. 51],

$$f \in L^p \Rightarrow T_{sf} \in L^p.$$  

Now using the analysis similar to that of [2], we can show that

$$T_{Ht}c(t) = \int_0^\pi f_c(u)\frac{du}{2 \tan u/2} + \beta(t) \text{ for a.e. } t > 0$$

where $\sum_{n=0}^\infty (a_n/2n) \sin nt = \beta(t) \in C$.

Hence, by [6, Theorem 2],

$$f \in L^\infty \Rightarrow f_c \in L^\infty \Rightarrow T_{Ht}c \in L^\infty \Rightarrow T_{sf} \in L^\infty.$$  

Now using the closed graph theorem, we can easily show the continuity of the operator $T_s$.

Proof of Theorem 1. ($\Rightarrow$) Let $\sigma_n f(x) = \sigma_n(x; f)$. Define

$$e_n(x) \equiv \sigma_n f(x) - [\sigma_n f(0)] \cos(n + 1)x.$$  

Then

$$\|e_n - f\|_\infty \leq \|\sigma_n f - f\|_\infty + |\sigma_n f(0)|.$$  

But $f(0) = 0 \Rightarrow |\sigma_n f(0)| \to 0$ as $n \to \infty$. Therefore $\|e_n - f\|_\infty \to 0$, and, hence,

$$\|T_s e_n - T_{sf}\|_\infty \to 0 \text{ as } n \to \infty.$$  

But $T_s e_n$ is continuous for all $n$, being a polynomial of degree $n$. Thus $T_{sf} \in C$.

($\Rightarrow$) Take $F = f - f(0)$. Then

$$T_s F(t) = T_{sf}(t) - f(0)(\text{sgn } t)(\pi - |t|)/2 \text{ for a.e. } t.$$  

But $T_{sf}$ and $T_s F$ are continuous functions—the second one is continuous because $F(0) = 0$. Hence, $f(0) = 0$. Q.E.D.

Remark. Once $T_{sf}$ is continuous, it is not very hard to show that the Fourier series of $T_{sf}$ converges uniformly. We just have to consider the difference of Cesàro means and the partial sums of $T_{sf}$ and use the strong summability of Fourier series of $f$ at $0$.

Corollary 1.1. Let $f$ be a continuous function and $a_0$ the constant term of its Fourier expansion. Then for every $x$,

$$\lim_{t \to 0+} \frac{1}{\pi} \int_t^\infty \frac{\tilde{f}(x + u) - \tilde{f}(x - u)}{2 \tan u/2} du = f(x) - a_0.$$  

Proof. It is enough to prove this for $x = 0$. We can also assume that $f = f_c$.

Take $G = f - f(0)$. Then, by (1.1),

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\[ T_S G(t) = \int_t^\infty \frac{\tilde{f}_e(u)}{2\tan u/2} \, du + F(t) - f_e(0)(\pi - t)/2 \quad \text{for } \pi > t > 0 \]

and

\[ T_S G(-t) = -T_S G(t) \quad \text{for } \pi > t > 0. \]

But \( T_S G \in C \) by the theorem. Hence,

\[ \lim_{t \to 0^+} \frac{1}{\pi} \int_t^\infty \frac{\tilde{f}_e(u)}{2\tan u/2} \, du = f_e(0). \quad \text{Q.E.D.} \]

**Proof of Theorem 2.** \((\Leftarrow)\) Since \( f \) and \( T_S f \) are continuous by [4, Theorem 5], the necessity of the condition is obvious from Theorem 1.

\((\Rightarrow)\) We shall consider two cases.

Case 1. \( f \in \Lambda(1, p) \) where \( 1 < p < \infty \). By [3, Theorem 24], \( f(x) - f(0) = \int_0^\infty F(t) \, dt \) where \( F \in L^p \). Let \( f = \sum_{n=0}^\infty (a_n \cos nt + b_n \sin nt) \). Then

\[ F(t) = f'(t) - \sum_{n=0}^\infty (-na_n \sin nt + nb_n \cos nt). \]

So

\[ \sum_{n=1}^\infty na_n \sin nt \sim [F(-t) - F(t)]/2 \in L^p. \]

Hence, by [1, Theorem 2],

\[ \sum_{n=1}^\infty \left( \sum_{j=n}^\infty a_j \right) \cos nt \sim F^* \in L^p. \]

But \( \sum_{n=0}^\infty a_j = f(0) = 0 \). Hence

\[ -\sum_{n=0}^\infty S_n(0; f) \cos nt \sim F^* \in L^p. \]

Now, by [3, Theorem 22], \( T_S f \in \Lambda(1, p) \).

Case 2. \( f \in \Lambda(\alpha, p) \) where \( 0 < 1/p < \alpha < 1 \). Define \( \sigma_n f \) as before and \( \sigma_{n,j} \) by

\[ \sigma_{n,j}(x) = \sum_{i=0}^j \left[ 1 - i/ (n + 1) \right] A_i(x; f). \]

By [7, Theorem 6], \( \|f - \sigma_n f\|_p = O(n^{-\alpha}) \). Now \( T_S \sigma_n f = P_n + Q_n \) where

\[ P_n(t) = \sum_{j=1}^n \left[ \frac{\sigma_{n,j}(0)}{j} \right] \sin jt \quad \text{and} \quad Q_n(t) \sim \left[ \sigma_{n,j}(0) \right] \sum_{j=n+1}^\infty \frac{(\sin jt)}{j}. \]

By [4, Theorem 5], \( f \in \Lambda(\alpha - 1/p) \). Hence, by [9, Vol. I, p. 123], \( |\sigma_n f(0)| = O(n^{-\alpha+1/p}) \). Now

\[ \sum_{j=1}^\infty (\sin jt)/j \sim (\sgn t)(\pi - |t|)/2 \in \Lambda(1/p, p). \]

Hence, by [7, Theorem 5], \( \|Q_n/\sigma_n f(0)\|_p = O(n^{-1/p}) \). Using Lemma 1.1 and the above relations, we get...
\[ \| T_S f - P_n \|_p \leq \| T_S f - T_S \sigma_n f \|_p + \| T_S \sigma_n f - P_n \|_p \leq A_p \| f - \sigma_n f \|_p + \| Q_n \|_p = O(n^{-\alpha}). \]

Hence, by [7, Theorem 2], \( T_S f \in \Lambda(\alpha, p) \). Q.E.D.

**Remarks.**

(i) Using the above technique, we can easily show that \( f \in \Lambda(\alpha) \) and \( f(0) = 0 \Rightarrow T_S f \in \Lambda(\alpha) \) where \( 0 < \alpha < 1 \).

(ii) Using Corollary 2.1 of [8], we can also show that \( f \in \Lambda(1, 1) \Rightarrow T_S f \in \Lambda(1, 1) \). But the corresponding statement for the class \( \Lambda(1/p, p) \), where \( p > 1 \), is not true as can be seen by taking \( f \sim \sum_{n=1}^{\infty} (\cos nx)/n \).

Imposing the condition \( |\sigma_n f(0)| = O(1) \), and using the above technique, one can prove that \( f \in \Lambda(1/p, p) \) implies \( T_S f \in \Lambda(1/p, p) \).

**References**


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