NUMERICAL RANGE OF A WEIGHTED SHIFT WITH PERIODIC WEIGHTS

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Abstract. Calculation of the numerical range of a weighted shift is reduced to the solution of a polynomial equation when the weights form a periodic sequence, or approach a periodic sequence from below.

Introduction. A weighted shift on $l^2$ or $l_+^2$ is a linear operator $S$ defined by $S e_n = s_n e_{n+1}$ where $(e_n)$ is an orthonormal basis, and $(s_n)$ a sequence of complex numbers. The numerical range of an operator $S$ is the set of complex numbers $(Sx, x)$ where $\|x\| = 1$; this is denoted $W(S)$. For definiteness we assume here a one-sided shift, indexed by positive integers; the proofs and results are the same for a two-sided shift.

Then we are given

$$x = x_1 e_1 + x_2 e_2 + \cdots, \quad x_i \text{ complex,} \quad \sum |x_i|^2 < \infty;$$

$$Sx = x_1 s_1 e_2 + x_2 s_2 e_3 + \cdots.$$

We begin with some simple facts about weighted shifts [1].

(1) $S$ is a bounded operator if and only if $(s_n)$ is a bounded sequence, and then $\|S\| = \sup |s_n|$.

(2) $S$ is unitarily equivalent to a shift with weights $t_i$ whenever $|t_i| = |s_i|$ for all $i$. In particular, $S$ is unitarily equivalent to $cS$ whenever $|c| = 1$.

(3) Therefore $W(S)$ has circular symmetry about 0: $cW = W$ whenever $|c| = 1$.

(4) Since $W$ is convex, it follows that $W(S)$ is a disk centered at 0; its radius $w(S)$ is the numerical radius of $S$.

It is an easy exercise to find $W(S)$ in some special cases. For example:

(5) If $|s_n| \leq K$ for all $n$, and $|s_n| \to K$, then $W(S) = K$.

By (2) it suffices to consider shifts with real nonnegative weights, $s_n \geq 0$, and we shall do so.

Theorem 1. If $(s_n)$ is a periodic sequence, of period $r$, then

$$w(S) = \max \{s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1 : x_i \text{ real,} \quad x_1^2 + \cdots + x_r^2 = 1\},$$

and finding this is equivalent to solving a polynomial equation of degree $r$.
Proof. First consider a sequence \( x \) consisting of the finite sequence of complex numbers \( \{x_1, x_2, \ldots, x_r\} \) repeated \( k \) times, with 0's thereafter. Then

\[
S_x = \{0, [s_1 x_1, s_2 x_2, \ldots, s_r x_r], \text{repeated } k \text{ times}, 0, 0, \ldots\},
\]

\[
(S_x, x) = k(s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1) - s_r x_r x_1,
\]

\[
(x, x) = k(\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_r\|^2),
\]

and for large \( k \) we see that \( (S_x, x)/(x, x) \) can be made arbitrarily close to

\[
\frac{s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1}{\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_r\|^2}.
\]

Therefore \( w(S) \) is at least equal to

\[
\max\{|s_1 x_1 x_2 + \cdots + s_r x_r x_1|: x_i \text{ complex, } \|x_1\|^2 + \cdots + \|x_r\|^2 = 1\}.
\]

By multiplying \( x_k \) by \( e^{i\theta_k} \) we may make these components real and nonnegative: this gives the problem:

Maximize \( s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1 \)

subject to \( x_1^2 + \cdots + x_r^2 = 1, s_k, x_k \) real.

The use of Lagrange multipliers gives the system:

\[
\begin{align*}
s_r x_r + s_1 x_2 &= \lambda x_1 \\
s_1 x_1 + s_2 x_3 &= \lambda x_2 \\
\vdots \\
s_{r-1} x_{r-1} + s_r x_1 &= \lambda x_r.
\end{align*}
\]

Elimination of the \( x_i \) gives a polynomial equation in \( \lambda \) of degree \( r \); \( x_2, \ldots, x_r \) are found by substitution (in terms of \( x_1 \)), and \( x_1 \) is then found by the relation

\[
x_1^2 + \cdots + x_r^2 = 1.
\]

We now establish that \( w(S) \) is no greater than this maximum value of \( s_1 x_1 x_2 + \cdots + s_r x_r x_1 \).

Lemma. If \( a_k, b_k \) are nonnegative constants with \( b_k \neq 0 \), then

\[
\frac{a_1 + a_2 + \cdots}{b_1 + b_2 + \cdots} \leq \sup_k \frac{a_k}{b_k}
\]

whenever the left side is defined.

Proof. We first show this for finite sums. If \( a/c \geq b/d \), then

\[
\frac{a + b}{c + d} \leq \frac{a + ad/c}{c + d} = \frac{a}{c}
\]

and the result for finite sums follows by induction. Then

\[
\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leq \max_{k \leq n} \frac{a_k}{b_k} \leq \sup_k \frac{a_k}{b_k}.
\]
and hence the lim sup of the left side satisfies the same inequality; this proves
the lemma.

Resuming the proof of Theorem 1: suppose \(|x| = 1\), and write the compo-
ents of \(x\) as

\[
x = \{a_1, a_2, \ldots, a_r; a_{21}, a_{22}, \ldots, a_{2r}; \ldots\},
\]

\(a_j\) complex. Then

\[
\frac{\langle Sx, x \rangle}{\langle x, x \rangle} = \frac{|s_1 a_1 a_2 + \cdots + s_r a_1 a_{21} + s_1 a_2 a_2 + \cdots + s_r a_2 a_{21} + \cdots|}{|a_1|^2 + \cdots + |a_r|^2 + |a_{12}|^2 + \cdots + |a_{2r}|^2 + \cdots}
\]

\[
\leq \sup_k \frac{|s_1 a_k a_{k+1} + \cdots + s_r a_1 a_{k+1} a_{k+1}|}{|a_k|^2 + \cdots + |a_{k+1}|^2}
\]

(1)

and

\[
\leq \sup_k \frac{|s_1 a_k a_{k+1} a_{k+2} + \cdots + s_r a_1 a_{k+1} a_{k+2}|}{|a_{k+1}|^2 + |a_{k+2}|^2 + \cdots + |a_{k+1}|^2}.
\]

(2)

Inequality (1) follows from the lemma, and (2) follows by deleting \(|a_1|^2\) from
the denominator (thus increasing the value of the fraction), regrouping terms
of the denominator, and applying the lemma.

Setting

\[
x_1 = \max(|a_k|, |a_{(k+1)}|), \quad x_j = |a_k| \quad \text{for } j = 2, \ldots, r,
\]

we see that

\[
\frac{\langle Sx, x \rangle}{\langle x, x \rangle} \leq \max\{s_1 x_1 x_2 + \cdots + s_r x_r x_1: x_1^2 + \cdots + x_r^2 = 1, x, \text{ real}\},
\]

and therefore \(w(S)\) is equal to this maximum value; this completes the proof
of Theorem 1.

**Theorem 2.** If \(s_k \leq p_k\) and \(s_k - p_k \to 0\) as \(k \to \infty\), where \(\{p_k\}\) is a periodic
sequence with period \(r\), then

\[
w(S) = \max\{p_1 x_1 x_2 + \cdots + p_r x_r x_1: x_1^2 + \cdots + x_r^2 = 1, x, \text{ real}\}.
\]

**Proof.** Given \(\epsilon > 0\), letting \(T\) be the shift with weights \(p_k\), let \(x\) be a unit
vector such that \((Tx, x) > w(T) - \epsilon\). Choose \(n\) such that \(|s_k - p_k| < \epsilon\) for
\(k \geq n\). Let \(y\) be the unit vector with \(y_k = 0, k = 1, \ldots, n; y_{k+n} = x_k, k = 1, 2, \ldots\). Then \((Ty, y) = (Tx, x) > w(T) - \epsilon\).

Now

\[
\|Ty - Sy\| \leq \sup_{k > n} |p_k - s_k| \leq \epsilon
\]

so \(|(Ty, y) - (Sx, y)| < \epsilon\) and so \((Sy, y) > w(T) - 2\epsilon\).

Therefore \(w(S) > w(T)\). Since \(0 \leq s_k \leq p_k\), we easily have \(w(S) \leq w(T)\),
and so the two are equal. By Theorem 1,

\[
w(T) = \max\{p_1 x_1 x_2 + \cdots + p_r x_r x_1: x_1^2 + \cdots + x_r^2 = 1, x, \text{ real}\} = w(S);
\]

this proves Theorem 2.
EXAMPLES. (1) If \( r = 2 \) the weights are \( a, b, a, b, \ldots \); we are to maximize \((a + b)x_1x_2\), that is, maximize \(x_1x_2\) subject to \(x_1^2 + x_2^2 = 1\). The solution is \(x_1 = x_2 = 1/\sqrt{2}\), \((a + b)x_1x_2 = (a + b)/2\), and so the numerical radius is the average of the two weights.

(2) If \( r = 3 \) the weights are \( a, b, c, a, b, c, \ldots \); we must maximize \(ax_1x_2 + bx_2x_3 + cx_3x_1\) with \(x_1^2 + \cdots + x_3^2 = 1\); we have the system

\[\begin{align*}
ax_2 + cx_3 &= \lambda x_1, \quad ax_1 + bx_3 = \lambda x_2, \quad bx_2 + cx_1 = \lambda x_3,
\end{align*}\]

which (if \(x_1 \neq 0\)) leads to the cubic equation \(\lambda^3 - (a^2 + b^2 + c^2)\lambda - 2abc = 0\).

NOTES. (1) The numerical range is always a disk about 0, of positive radius except in the trivial case where all the weights are zero.

(2) If any weight is zero then the disk is closed; for (assuming \(s_r = 0\) for example) then \((Sx, x)/(x, x)\) is actually equal to the expression (1), which in turn attains its maximum on the compact sphere (2).

(3) For weights \((1, 1, 1, \ldots)\) the disk is open; for \(|(Sx, x)| = 1, |x| = 1\), would imply \(Sx = kx, |k| = 1\), which is impossible.

(4) I surmise, but have yet to prove, that the disk is open whenever all the weights are nonzero; that is, \((Sx, x)\) cannot attain its sup \(w(S)\) for \(|x| = 1\).

REFERENCE