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COUNTABLE PARACOMPACTNESS OF F_σ-SETS

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ABSTRACT. If each F_σ-set in $X \times Y$ is countably paracompact, then either X is normal or no countable discrete subset of Y has a limit point. It follows that, for each cardinal number $\mathfrak{m}$, there is an $\mathfrak{m}$-paracompact space containing a noncountably paracompact F_σ-subset.

In [2], Katětov shows that if $X \times Y$ is hereditarily normal then either X is perfectly normal or every countable subset of Y is closed. The techniques of [2] were employed in [4] to show that if $X \times Y$ is hereditarily countably paracompact then either X is perfectly normal or every countable discrete[1] subset of Y is closed. We use similar techniques to obtain the main theorem of this note.

Before proceeding, we state a lemma which was proved in [5].

LEMMA. *The space X is normal if and only if for each pair (H, K) of mutually exclusive closed sets there are sequences $\{U_i\}$ and $\{V_i\}$ for open sets such that*

$$H \subset \bigcap_{i=1}^{\infty} U_i \subset \bigcap_{i=1}^{\infty} \overline{U}_i \subset X - K, \tag{1}$$

and

$$K \subset \bigcap_{i=1}^{\infty} V_i \subset \bigcap_{i=1}^{\infty} \overline{V}_i \subset X - H. \tag{2}$$

The main result of this note is the following:

THEOREM 1. *If each F_σ-set in $X \times Y$ is countably paracompact, then either X is normal or no countable discrete subset of Y has a limit point.*

PROOF. Suppose there is a countable discrete subset $A = \{y_1, y_2, \ldots\}$ of Y that has a limit point y_0 that is not in A and let (H, K) be a pair of mutually exclusive closed sets in X. For each $i > 0$, let $F_i = \{(x,y) \in X \times Y | y = y_i\}$ and let $F_0 = \{(x,y) \in X \times Y | x \in H, y = y_0\}$. Then $F = \bigcup_{i=0}^{\infty} F_i$ is an F_σ-set in $X \times Y$. Let $W_0 = \{(x,y) \in F | x \notin K\}$. For each $i > 0$, let $W_i = F_i$. Then $\{W_0, W_1, \ldots\}$ is a countable collection of sets, open in F, that covers F. Since F is countably paracompact, there is a countable collection of sets $\{W_0', W_1', \ldots\}$, open in F, covering F such that (a) $W_i' \subset W_i$ for each $i \geqq 0$ and (b) $\{W_0', W_1', \ldots\}$ is locally finite at each point of F. For each $i > 0$, let

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[1] A set is discrete if it has no limit point in itself.

$V_i = \Pi_X W_i'$, where Π_X is the projection of $X \times Y$ onto X. Then $\{V_i\}$ is a sequence of open sets in X each member of which contains K. Let $h \in H$. We will show that there is an i so that $h \notin \overline{V_i}$. To this end, let S and T be open sets in X and Y, respectively, so that (a) $h \in S$ and $y_0 \in T$ and (b) $S \times T$ meets only finitely many members of $\{W_0', W_1', W_2', \ldots\}$. Thus, there is an N such that $W_N' \cap (S \times T) = \varnothing$ and $y_N \in T$; and so $h \notin \overline{V_N}$. It follows that we have a sequence $\{V_i\}$ such that (2) is true. In an analogous fashion, we can obtain a sequence of open sets in X, $\{U_i\}$, such that (1) is true. That X must be normal follows from our Lemma.

The following theorem was proved by K. Morita [3].

THEOREM 2. *If X is normal and countably paracompact, then each F_σ-set in X is countably paracompact.*

From Theorems 1, 2, and 4 of [1], we have the following:

COROLLARY 1. *Each F_σ-set in $X \times [0, 1]$ is countably paracompact if and only if X is normal and countably paracompact.*

COROLLARY 2. *For each cardinal $\mathfrak{m}$, there is an $\mathfrak{m}$-paracompact space that contains an F_σ-set that is not countably paracompact.*

PROOF. Let $\mathfrak{m}^+$ be first cardinal greater than $\mathfrak{m}$. Let α be the ordinal $\mathfrak{m}^+ \cup \{\mathfrak{m}^+\} = \mathfrak{m}^+ + 1$. Then $(\alpha \times \alpha) - \{(\{\mathfrak{m}^+\}, \{\mathfrak{m}^+\})\}$ is $\mathfrak{m}$-paracompact but not normal.

COROLLARY 3. *The following conditions are equivalent for the topological space X:*

(1) *X^w is normal.*

(2) *X^w is normal and countably paracompact.*

(3) *Each F_σ-set in X^w is countably paracompact.*

PROOF. We need only prove that (1) and (3) are equivalent. (1) implies (3): Suppose $X^w = X^w \times X^w$. Assuming that X has at least two points, then there is a countable discrete subset of X^w with a limit point. It follows from Dowker's Theorem that X^w is normal and countably paracompact.

In a similar fashion, we can apply our Theorem 1 to show that (3) implies (1).

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