ON THE SPACE OF PIECEWISE LINEAR HOMEOMORPHISMS OF A MANIFOLD

ROSS GEOGHEGAN AND WILLIAM E. HAVER

Abstract. Let $M^n$ be a compact PL manifold, $n \neq 4$; if $n = 5$, suppose $\partial M$ is empty. Let $H(M)$ be the space of homeomorphisms on $M$ and $H^\ast(M)$ the elements of $H(M)$ which are isotopic to PL homeomorphisms. It is shown that the space of PL homeomorphisms, $PLH(M)$, has the finite dimensional compact absorption property in $H^\ast(M)$ and hence that $(H^\ast(M), PLH(M))$ is an $(l_2, l_2^f)$-manifold pair if and only if $H(M)$ is an $l_2$-manifold. In particular, if $M^2$ is a 2-manifold, $(H(M^2), PLH(M^2))$ is an $(l_2, l_2^f)$-manifold pair.

1. Introduction. Let $M$ be a compact piecewise linear (PL) manifold, possibly with boundary. We shall study the pair $(H(M), PLH(M))$, where $H(M)$ denotes the space of all homeomorphisms of $M$ onto itself and $PLH(M)$ the subspace of all piecewise linear homeomorphisms. All function spaces will be assumed to have the compact-open topology.

For some years now there has been considerable interest in the question of whether $H(M)$ is an $l_2$-manifold; i.e., a separable metric space which is locally homeomorphic to $l_2$, the hilbert space of square-summable sequences. For an arbitrary compact manifold $M$, it is known that $H(M)$ is uniformly locally contractible (Chernavskii [6] and Edwards and Kirby [8]) and that $H(M) \times l_2$ is homeomorphic to $H(M)$ (Geoghegan [11]). By a theorem of Toruńczyk [20], $H(M)$ is an $l_2$-manifold if and only if $H(M)$ is an ANR. When $M$ is a 2-manifold, Luke and Mason [18] have shown that $H(M)$ is an ANR (hence an $l_2$-manifold). But it is still unknown whether or not $H(M^n)$ is an ANR when $n > 2$.

In the PL case more is known. $PLH(M)$ is the countable union of finite-dimensional compacta (Geoghegan [12]), and is uniformly locally contractible (Edwards, see [13] and Gauld [10]). Haver [13] has shown that any such space is an ANR. Toruńczyk [20] has shown that any such ANR becomes an $l_2^f$-manifold when multiplied by $l_2^f$ (where $l_2^f$ denotes the subspace of $l_2$ consisting of those sequences having only finitely many nonzero entries). Hence $PLH(M) \times l_2^f$ is an $l_2^f$-manifold. Finally, Keesling and Wilson [16] have shown that $PLH(M) \times l_2^f$ is homeomorphic to $PLH(M)$. Hence $PLH(M)$ is an $l_2^f$-manifold.

In general it is not true that $PLH(M)$ is dense in $H(M)$. For example, Kirby and Siebenmann (see [17]) have shown that $H(S^2 \times S^3)$ has a component containing no PL homeomorphism. They have also shown that if the
third cohomology group of $M^n$ with $\mathbb{Z}_2$-coefficients is trivial, and if $n \geq 6$, or $n = 5$ and $\partial M$ is empty, $PLH(M^n)$ is dense in $H(M^n)$. In low dimensions, $PLH(M^n)$ is always dense in $H(M^n)$: this is obvious when $n = 1$; see Rado [19] when $n = 2$; see Bing [4] when $n = 3$.

For our purposes we can avoid reference to the difficult work of Kirby and Siebenmann. Using earlier work of Connell [7] we can prove

**Theorem 1.** Let $M^n$ be a compact PL manifold, $n \neq 4$; if $n = 5$ suppose $\partial M$ is empty. Then the closure of $PLH(M)$ in $H(M)$ is the union of components of $H(M)$.

Let $H^*(M)$ be the subset of $H(M)$ consisting of those homeomorphisms which are isotopic to PL homeomorphisms. The above remarks show that $H^*(M)$ is often equal to $H(M)$. Theorem 1 says that [with certain dimension restrictions] $PLH(M)$ is always dense in $H^*(M)$.

A pair of spaces $(X, X')$ is an $(l_2, l_{\{})$-manifold pair if $X$ is an $l_2$-manifold, and if there exist an open cover $\mathcal{U}$ of $X$ and open embeddings $\{f_U: U \to l_2 \ | \ U \in \mathcal{U}\}$ such that for each $U, f_U(U \cap X') = f_U(U) \cap l_{\{}$. In other words $X'$ sits in $X$, locally, as $l_{\{}$ sits in $l_2$.

**Theorem 2.** Let $M^n$ be a compact PL manifold, $n \neq 4$; if $n = 5$, suppose $\partial M$ is empty. Then $(H^*(M^n), PLH(M^n))$ is an $(l_2, l_{\{})$-manifold pair if and only if $H(M)$ is an $l_2$-manifold.

Theorem 2 is proved by showing that $PLH(M^n)$ has the "finite-dimensional compact absorption property" in $H^*(M^n)$: see §3 for the definition. This theorem, when combined with our opening remarks, yields

**Corollary 1.** If $M^2$ is a compact PL 2-manifold, $(H(M^2), PLH(M^2))$ is an $(l_2, l_{\{})$-manifold pair.

Theorem 1 combined with the fact that each of $PLH(M)$ and $H(M)$ is uniformly locally contractible immediately gives:

**Corollary 2.** Let $M^n$ be as in Theorem 2. Then the inclusion $PLH(M^n) \to H^*(M^n)$ is a weak homotopy equivalence (a homotopy equivalence if $n = 2$).

**Corollary 3.** The inclusion of the identity component of $PLH(M^n)$ into the identity component of $H(M^n)$ is a weak homotopy equivalence (a homotopy equivalence if $n = 2$).

We will let $H_0(I^n)$ denote the set of elements of $H(I^n)$ which equal the identity when restricted to the boundary of $I^n$ and $PLH_0(I^n) = PLH(I^n) \cap H_0(I^n)$. If $X$ is any space, let $1_X$ denote the identity homeomorphism on $X$. If $f \in H(M)$, let $N_\epsilon(f) = \{ h \in H(M) \mid d(h, f) < \epsilon \}$ where $d$ is a metric on $M$.

2. **Proof of Theorem 1.** As explained in §1, the low-dimensional cases are well known.

**Lemma 1.** If $n > 4$, $PLH_0(I^n)$ is dense in $H_0(I^n)$.

**Proof.** Let $H_0'(I^n)$ be the group of homeomorphisms of $I^n$ which fix a neighborhood of $\partial I^n$. More precisely.
$H_\delta'(I^n) = \{ h \in H_\delta(I^n) | \text{for some } 0 < a < 1, \}
\quad h(x) = x \text{ whenever } d(x,0) \geq a \}.$

By [9, Theorem 3], $H_\delta'(I^n)$ is a simple subgroup of $H_\delta(I^n)$. $H_\delta'(I^n)$ is clearly dense in $H_\delta(I^n)$. Consider

$$H_\delta"(I^n) = \{ h \in H_\delta'(I^n) | \text{if } T \text{ is a PL structure on } I^n \text{ and if } \epsilon > 0, \}
\quad \text{there exists } f \in H_\delta'(I^n) \text{ such that } f \text{ is PL relative to } T \text{ and } d(f,h) < \epsilon \}.$$

**Claim (following Connell).** $H_\delta"(I^n)$ is normal in $H_\delta'(I^n)$.

**Proof of Claim.** Let $h \in H_\delta"(I^n)$ and $g \in H_\delta'(I^n)$. We must show $g^{-1}hg \in H_\delta"(I^n)$. Let $T$ be a PL structure on $I^n$ and let $\epsilon > 0$ be given. There exists $\delta > 0$ such that for all $x, y \in I^n$, $d(x,y) < \delta$ implies $d(g^{-1}(x),g^{-1}(y)) < \epsilon$. Let $T'$ be the PL structure on $I^n$ which is the image of $T$ under $g$. Since $h \in H_\delta"(I^n)$ there exists $f \in H_\delta'(I^n)$ which is PL with respect to $T'$ and satisfies $d(h,f) < \delta$. Thus $d(g^{-1}hg,g^{-1}fg) < \epsilon$. Now $g^{-1}fg$ is PL with respect to $T$. It follows that $g^{-1}hg \in H_\delta"(I^n)$; the Claim is proved.

Since $H_\delta'(I^n)$ is dense and simple, it only remains to show that $H_\delta"(I^n)$ contains some $h_0 \neq 1_{I^n}$. When $n > 6$, Connell shows on p. 331 of [7] that any nontrivial symmetric radial expansion of $I^n$ which fixes a neighborhood of $\partial I^n$ can serve as $h_0$. The restriction $n > 6$ arises from the radial engulfing lemma used in [7]. But in [3], Bing proves a stronger engulfing lemma which makes Connell's proof valid for $n > 4$; see [3, p. 3] and [7, p. 337].

**Proof of Theorem 1.** Let $H$ be a component of $H(M)$. We show that $H \cap \text{PLH}(M)$ is either dense in $H$ or is empty.

We first show the PL homeomorphisms are dense in a neighborhood of $1_M$: i.e., that there is a $\delta > 0$ such that whenever $g \in N_\delta(1_M)$ and $\epsilon > 0$ are given, there exists $h \in \text{PLH}(M) \cap N_\delta(g)$.

Choose an open cover $(B_1, \ldots, B_p)$ of $M$ so that for each $i$, $B_i$ is a PL $n$-ball and so that $D_i = \overline{B}_i \cap \partial M$ is either empty or a PL $(n-1)$-ball. In [8, p. 19] it is shown that there is a $\delta > 0$ so that each $g \in N_\delta(1_M)$ can be written as the composition $g = g_p \cdots g_2 \cdot g_1$ of $p$ homeomorphisms such that each $g_i$ is supported by $B_i$. Since each $g_i$ is uniformly continuous, there exist positive numbers $\eta_1, \ldots, \eta_p$ such that if for each $i$ the homeomorphism $h_i$ satisfies $d(h_i, g_i) < \eta_i$, then $d(h_i, g_p \cdots h_2 \cdot \cdot \cdot h_1, g_1) < \epsilon$.

If $\overline{B}_i \cap \partial M = \emptyset$, then $g_i|\partial B_i$ lies in $H_\delta(\partial B_i)$ and, by the previous lemma, there exists a PL homeomorphism $h_i \in H_\delta(\partial B_i)$ such that $d(h_i, g_i|\partial B_i) < \eta_i$. Extend $h_i$ by the identity to form $h_i \in \text{PLH}(M)$ and note that $d(h_i, g_i) < \eta_i$. If $\overline{B}_i \cap \partial M = D_i \neq \emptyset$ (and is therefore a PL $(n-1)$-ball), $g_i$ maps $D_i$ homeomorphically onto itself and fixes $\partial D_i$. By the previous lemma there is a PL homeomorphism $f_i \in H_\delta(D_i)$ approximating $g_i|D_i$. Extending $f_i$ by the identity we get a PL homeomorphism $\tilde{f}_i : \partial B_i \to \partial B_i$ which approximates $g_i|\partial B_i$. Thus $\tilde{f}_i g_i^{-1}|\partial B_i$ approximates $1_{\partial B_i}$. Coning at a point in the interior of $B_i$, we obtain a homeomorphism $F_i$ of $\overline{B}_i$ which agrees with $\tilde{f}_i g_i^{-1}$ on $\partial \overline{B}_i$ and which approximates $1_{\overline{B}_i}$. Let $g_i' = F_i g_i^{-1} : \overline{B}_i \to \overline{B}_i$. Note that $g_i'$ approximates $g_i$ and is PL on $\partial \overline{B}_i$. Omitting epsilonics, we may say that $d(g_i', g_i|B_i) < \eta_i/2$. Note
that $g_i'$ fixes $\partial B_i \cap \text{int} D_i$. By coning, construct a PL homeomorphism $g_i''_i : \overline{B}_i \to \overline{B}_i$ which extends $g_i'|\partial B_i$. Then $g_i''_i(g_i''_i)^{-1} \in H_3(B_i)$, so there exists $\tilde{g}_i \in PLH_3(B_i)$ such that $d(\tilde{g}_i,g_i''_i(g_i''_i)^{-1}) < \eta_i/2$. Extend $\tilde{g}_i g_i''_i$ by the identity to $h_i \in PLH(M)$ and note that $d(h_i,g_i) < \eta_i$. The required $h$ is $h_p \cdots h_1$.

Now let $H$ be a component of $H(M)$ such that $H \cap PLH(M) \neq \emptyset$. Let $A = \text{cl}_H (H \cap PLH(M))$. We will show that $A$ is open in $H$, from which it will follow that $A = H$. Let $f \in A$. Let $\delta$ be as in the first part of the proof, and assume without loss of generality that $N_{2\delta}(f) \subset H$. The local connectedness of $H(M)$ is used here. Let $g_0 \in N_{\delta}(f)$. We must show that $g_0 \in A$. Let $\eta > 0$ be given; we may assume $\eta \leq \delta$. Since $d(g_0 f^{-1},1_M) < \delta$, there exists $h \in PLH(M)$ such that $d(h,g_0 f^{-1}) < \eta/2$ and hence $d(hf,g_0) < \eta/2$. Since $h$ is uniformly continuous, there exists $\gamma > 0$ such that $d(h(x),h(y)) < \eta/2$ whenever $d(x,y) < \gamma$. Choose $f' \in PLH(M)$ such that $d(f',f) < \gamma$. Then $hf' \in PLH(M)$ and $d(hf',g_0) \leq d(hf',hf) + d(hf,g_0) < \eta$. Therefore $hf' \in N_{\eta}(g_0) \cap H$. Since $hf'$ is PL, $g_0 \in A$.

3. Proof of Theorem 2. A subset $X'$ of a metric space $X$ is said to have the finite-dimensional compact absorption property (f.d. cap) if $X' = \bigcup_{n=1}^{\infty} A_n$ where (i) each $A_n$ is finite dimensional and compact, and (ii) given a finite-dimensional compactum $A$, a closed subset of $B$ of $A$, an embedding $f : A \to X$ such that $f(B) \subset X'$, and a number $\epsilon > 0$, there exists an embedding $h : A \to X$ such that $d(f,h) < \epsilon$, $h(A) \subset X'$ and $h(b) = f(b)$ whenever $b \in B$.

This property characterizes $(l_2,l_2)$-manifold pairs: precisely

**Proposition 1 (West [21]).** Let $(X,X')$ be a pair of metric spaces such that $X'$ has the f.d. cap in $X$. Then $(X,X')$ is an $(l_2,l_2)$-manifold pair if and only if $X$ is an $l_2$-manifold.

Where there is no confusion, we will not distinguish between a complex $P$, and its underlying point set, $|P|$. Let $PLH(M)$ be the closure of $PLH(M)$ in $H(M)$.

**Lemma 2.** Let $P$ be a finite complex and $B$ a closed subset of $P$. Suppose $f : P \to \overline{PLH(M)}$ is a map such that $f(B) \subset PLH(M)$ and $f(P \setminus B) \cap f(B) = \emptyset$. Then given $\epsilon > 0$, there is a map $g : P \to PLH(M)$ such that $d(f,g) < \epsilon$, $g|B = f|B$ and $g(P \setminus B) \cap g(B) = \emptyset$.

**Proof.** Let $n = \dim P$. Whenever $W$ is a closed subset of $P \setminus B$ let $
eta_W = 1/4 \min(\epsilon,d(f(W),f(B)))$.

Using the fact that $PLH(M)$ is uniformly locally contractible, form a triangulation $T$ of $P \setminus B$ satisfying:

(a) for each simplex $\sigma$ of $T$ there is a sequence $0 = \delta_{\sigma,0} < \delta_{\sigma,1} < \cdots < \delta_{\sigma,n} \leq \eta_{\sigma}$ such that any subset of $PLH(M)$ of diameter $< 3\delta_{\sigma,i}$ is contractible in a subset of $PLH(M)$ of diameter $< \delta_{\sigma,i+1}$, $0 \leq i \leq n - 1$;

(b) $\text{diam} f(\sigma) < \delta_{\sigma,0}$. (First choose a triangulation $\hat{T}$, next choose the $\delta_{\sigma,i}$'s with reference to $\hat{T}$ to satisfy (a); then subdivide $\hat{T}$ to get $T$ satisfying (b).)

Define $g : T^0 \to PLH(M)$ as follows. If $v$ is a vertex of $T^0$, let $g(v)$ be any
point of $PLH(M)$ such that $d(g(v), f(v)) < \delta_{a,0}$ for each of the (finite number of) simplexes $a$ of $T$ which contains $v$ as a vertex.

We next define $g: T^1 \to PLH(M)$. Let $\tau$ be a 1-simplex of $T^1$ and let $\sigma_0 < T$ be a simplex of which $\tau$ is a face. Then $\text{diam} f(\tau) \leq \text{diam} f(\sigma_0)$. Hence $\text{diam} g(\partial \tau) < 3\delta_{\sigma_0,0}$. We can therefore define

$$
g|\partial \tau: \tau \to PLH(M)$$

extending $g|\partial \tau$ in such a way that $\text{diam} g(\tau) < \delta_{a,1}$ for each of the (finite number of) simplexes $a$ of $T$ which contains $\tau$ as a face.

Assume, inductively, that we have defined $g|T^j: T^j \to PLH(M)$, such that if $\tau$ is a $j$-simplex of $T$, then $\text{diam} g(\tau) < \delta_{a,j}$ for each of the (finite number of) simplexes $a$ of $T$ which contains $\tau$ as a face.

Let $\tau$ be a $(j + 1)$-simplex of $T$. Note that $\text{diam} g(\partial \tau) < 3\delta_{a,j}$ for each simplex $a$ of $T$ which contains $\tau$ as a face. Therefore $g|\partial \tau$ can be extended to $\tau$ in such a way that $\text{diam} g(\tau) < \delta_{a,j+1}$ for each simplex $a$ of $T$ of which $\tau$ is a face.

Thus, by induction we define a continuous map $g: T \to PLH(M)$ so that if $\tau$ is a simplex of $T$, $\text{diam} g(\tau) < \eta$. Finally define $g$ on $B$ to agree with $f$.

We now show that $d(f, g) < \epsilon$. If $p \in B$, $d(f(p), g(p)) = 0$. If $p \in P \setminus B$ let $\tau$ be a simplex of $T$ containing $p$: $d(f(p), g(p)) \leq 3\eta < \epsilon$.

To show that $g(P \setminus B) \cap g(B) = \emptyset$, we observe that if $p \in P \setminus B$ and $p$ lies in the simplex $\tau$ of $T$,

$$
d(g(p), g(B)) \geq d(f(p), f(B)) - d(f(p), g(p))$$

$$
\geq d(f(p), f(B)) - 3\eta > 0.
$$

Finally we show that $g$ is continuous on $B$. Let $g \in B$ and $\eta > 0$ be given. Choose $\delta > 0$ so that if $d(p, q) < \delta$, $d(f(p), f(q)) < \eta/2$. Assume $p \in P \setminus B$. Then

$$
d(g(p), g(q)) \leq d(g(p), f(p)) + d(f(p), f(q))$$

$$
\leq d(f(\tau), f(B)) + d(f(p), f(q)) \leq 2d(f(p), f(q)) < \eta.
$$

Let $s$ denote the countable infinite product of open unit intervals and $s_f = \{\{x_i\} \in s \mid \text{for at most finitely many } i, x_i \neq 0\}$ with metric $d(x, y) = \sum_i \frac{1}{2^i} |x_i - y_i|$.

**Lemma 3.** Let $P$ be a finite complex, $B$ a closed subset of $P$, $V$ an open subset of $s_f$. Let $g: P \to V$ be a continuous map. Then given $\epsilon > 0$, there is a map $h: P \to V$ such that $h|B = g|B$, $d(g, h) < \epsilon$, $h(B) \cap h(P \setminus B) = \emptyset$ and $h|P \setminus B$ is injective.

**Proof.** This is standard infinite-dimensional topology: we sketch the proof, leaving epsilomics to the reader. Let $p_m: s_f \to s_f$ be the projection onto the first $m$ coordinates. If $m$ is large, $g$ is homotopic in $V$ to $p_m g$ by a small homotopy. If $m$ is large enough, $p_m g$ is homotopic in $V$ to an embedding $h': P \to V$ by a small homotopy. $h' g^{-1}|g(B)$ is a homeomorphism of $g(B)$ onto $h'(B)$ which is homotopic to the identity by a small homotopy. By Theorem 2.25 of [5], there
is a homeomorphism $k$ of $s^r$ close to the identity, extending $h'g^{-1}|g(B)$. The required $h$ is $k^{-1}h': P \to V$.

**Proof of Theorem 2.** By Theorem 1, $H^*(M)$ is an $l^2$-manifold if and only if $H(M)$ is. By Proposition 1 it is enough to show that $PLH(M)$ has the f.d. cap in $H^*(M)$. By [12, Theorem 1.9], $PLH(M)$ is the countable union of finite-dimensional compacta. We must verify (ii) the f.d. cap.

Suppose we are given a finite-dimensional compactum $A$, a compact subset $B$, an embedding $f: A \to H^*(M)$ such that $f(B) \subset PLH(M)$, and $\epsilon > 0$. By Theorem 1, $PLH(M) = H^*(M)$. Since $H^*(M)$ is locally contractible, $f$ extends to a map $f': P \to H^*(M)$ where $P$ is a finite complex (see [15, p.150]). As explained in §1, $PLH(M)$ is an $l^2$-manifold. By Theorem 7 of [14], $PLH(M)$ is homeomorphic to an open subset of $l^2$, and hence to an open subset of $s^r$ since $l^2$ and $s^r$ are homeomorphic [1, 2]. Thus, Lemmas 2 and 3 give a map $h: P \to PLH(M)$ with $d(f',h) < \epsilon$, $hB' = f^{-1}B'$ where $B'^{-1} = f^{-1}(f(B))$, $h(B') \cap h(P \setminus B') = \emptyset$ and $h|P \setminus B'$ is injective. $h|A$ is the desired embedding.

**References**


DEPARTMENT OF MATHEMATICAL SCIENCES, STATE UNIVERSITY OF NEW YORK AT BINGHAMTON, BINGHAMTON, NEW YORK 13901

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916