ON REALIZING CENTRALIZERS OF CERTAIN ELEMENTS IN THE FUNDAMENTAL GROUP OF A 3-MANIFOLD

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ABSTRACT. The main result in this note is that if $\lambda$ is a simple loop in the boundary of a compact, irreducible, orientable 3-manifold $M$ and $[\lambda] \neq 1 \in \pi_1(M)$, one can represent the centralizer of $[\lambda]$ in $\pi_1(M)$ by a Seifert fibred submanifold of $M$.

Introduction. The main result in this note is a partial answer to a question of Jaco [2]. Jaco has shown in [2] that the centralizer of a nontrivial element in the fundamental group of a sufficiently large compact, orientable 3-manifold is isomorphic to the fundamental group of a Seifert fibre space. He also suggests that one might geometrically realize this group by a submanifold of the ambient manifold. It is the purpose of this note to show that the above realization can in fact be made if the element is represented by a simple loop in the boundary of the 3-manifold and the 3-manifold is irreducible.

Proposition 7.1 in [2] is quite similar to our Theorem 2. Our notation and definitions are standard unless otherwise indicated.

We say that a manifold $N$ is properly embedded in a manifold $M$ if $N \cap \partial M = \partial N$. Let $A$ be an annulus. A spanning arc $\alpha$ of $A$ is an arc properly embedded in $A$ such that $A - \alpha$ is simply connected. Throughout the remainder of this paper $\alpha$ will denote a spanning arc of $A$.

**Proposition 1.** Let $A_1, \ldots, A_m$ be a collection of annuli properly embedded in $M$ such that $A_i \cap A_j = \partial A_i = \partial A_j$ for $1 \leq j < i \leq m$. Let $f: (A, \partial A) \to (M, \partial M)$ be a map such that $(1) f(\partial A) = \partial A_1$, $(2) f_*: \pi_1(A) \to \pi_1(M)$ is monic, and $(3) f(\alpha)$ is not homotopic rel its boundary to an arc in $\bigcup_{i=1}^m A_i$. Then there is an embedding $g: (A, \partial A) \to (M, \partial M)$ such that

1. $g(A) \cap A_i = \partial A_i$ for $i = 1, \ldots, m$,
2. $g_*: \pi_1(A) \to \pi_1(M)$ is monic,
3. $g(\alpha)$ is not homotopic rel its boundary to an arc in $\bigcup_{i=1}^m A_i$.

**Proof.** We show first that $f|\partial A$ may be assumed to be an embedding. Let $C_1$ and $C_2$ be the components of $\partial A$. Let $(\tilde{M}, \rho)$ be the covering space of $M$ associated with the subgroup of $\pi_1(M)$ generated by the class of the simple loop $f(C_i)$. Since $f_*\pi_1(A) \subseteq \rho_*\pi_1(M)$, there is a map $\tilde{f}: (A, \partial A) \to (\tilde{M}, \partial \tilde{M})$ such that $\rho \tilde{f} = f$. It is a consequence of the theorem in [8] that there is an embedding $\tilde{f}_*: (A, \partial A) \to (\tilde{M}, \partial \tilde{M})$ such that $\tilde{f}_*(\partial A) = \tilde{f}(\partial A)$. We may...
suppose that \( \tilde{f}_1(\partial \alpha) = \tilde{f}(\partial \alpha) \). Since \( \pi_1(M) \) is generated by \( \tilde{f}_1(C_1) \), we may suppose that the loop formed by traversing first \( \tilde{f}_1(\alpha) \) and then \( \tilde{f}(\alpha) \) is nullhomotopic.

Since the path \( f(\alpha) \) is end-point fixed homotopic in \( M \) to the path \( \rho_1(f)(\alpha) \), one may assume that \( f(C_1) \) is an embedding of \( C_1 \) into \( \partial A_1 \). Using a similar argument, we may show that there is no loss in generality in assuming further that \( f(C_2) \) is also an embedding of \( C_2 \) into \( \partial A_1 \).

It is a consequence of Theorem 1' in [1] that there is an embedding \( g: (A, \partial A) \to (M, \partial M) \) such that \( g(\partial A) = \partial A_1 \) and \( g(\alpha) \) is not homotopic rel its boundary to an arc in \( \bigcup_{i=1}^{n} A_i \). After the usual general position and cutting arguments, we may suppose that \( g^{-1}(\bigcup_{i=1}^{n} A_i) \) is the union of a collection of disjoint simple essential loops \( \lambda_1, \ldots, \lambda_r \). Let \( A_1, \ldots, \overline{A}_{r-1} \) be the closures of the components of \( A - \bigcup_{i=1}^{r} \overline{A}_i \). We may suppose that \( \alpha \) meets each of the \( \overline{A}_i \) in a spanning arc \( \alpha_i \) for \( i = 1, \ldots, r - 1 \). Since \( g(\alpha) \) is not homotopic rel its boundary to an arc on \( \bigcup_{i=1}^{r} A_i \), there is a \( j \) such that \( g(\alpha_j) \) is not homotopic rel its boundary to an arc on \( \bigcup_{i=1}^{r} A_i \). Now it is easy to obtain the desired embedding by considering \( g|\overline{A}_j \). This completes the proof of Proposition 1.

Let \( G \) be a group and \( \sigma \in G \). We denote the centralizer of \( \sigma \) in \( G \), i.e. \( \{g \in G: \sigma g = g \sigma \} \) by \( C(\sigma) \). The following theorem gives a positive partial answer to a question posed by Jaco in [2].

**Theorem 1.** Let \( M \) be a compact, connected, irreducible, orientable 3-manifold, \( \lambda \) a simple loop in \( \partial M \) which is not nullhomotopic in \( M \), and \( y \) a point in \( \lambda \). Then there is a submanifold \( N \subseteq M \) such that

1. \( \pi_1(N, y) = C([\lambda]) \subseteq \pi_1(M, y) \);
2. \( N \) is a (possibly trivial) Seifert fibre space;
3. \( \partial N \cap \partial M \) is an annular neighborhood \( A^* \) of \( \lambda \);
4. \( \partial N - A^* \) is incompressible in \( M \).

**Proof.** It is a consequence of the loop theorem [4], [6] that \( [\lambda] \) is of infinite order in \( \pi_1(M, y) \). Let \( A \) be an annular neighborhood of \( \lambda \) in \( \partial M \). Let \( A_1, \ldots, A_n \) be a maximal collection of annuli properly embedded in \( M \) such that

1. \( \partial A_i = \partial A \) for \( i = 1, \ldots, n \);
2. \( A_i \cap A_j = \partial A_i \) for \( 1 \leq i < j \leq n \);
3. If \( \alpha_j \) is a spanning arc of \( A_j \) where \( 1 \leq j \leq n \), \( \alpha_j \) is not homotopic rel its boundary to an arc in \( \bigcup_{i \neq j} A_i \cup \overline{A} \).

It follows from the theorem on p. 60 in [7] that there are at most finitely many disjoint nonparallel annuli properly embedded in \( M \) and satisfying condition (1) above.

Suppose \( x \in C([\lambda]) \). Then we claim \( x \) has a representative loop on \( \bigcup_{i=1}^{n} A_i \cup \overline{A} \).

If our claim is false, we let \( T \) be a torus and \( \lambda_1 \) and \( \lambda_2 \) simple loops on \( T \) such that \( \lambda_1 \cap \lambda_2 \) is a single point and \( T - (\lambda_1 \cup \lambda_2) \) is simply connected. Since \( x \) and \( [\lambda] \) commute, there is a map \( \tilde{f}: T \to M \) carrying a neighborhood \( R \) of \( \lambda_1 \) homeomorphically onto \( A \) and \( \lambda_2 \) to a representative of \( x \) in \( \pi_1(M) \). One obtains an annulus \( A \) by removing the interior of \( R \) from \( T \) so that \( \tilde{f} \) induces a map \( f: (A, \partial A) \to (M, \partial A) \). We observe that if \( \alpha \) is a spanning arc of \( A \), \( f(\alpha) \) is not homotopic rel its boundary to an arc on \( \bigcup_{i=1}^{n} A_i \cup \overline{A} \), since \( x \)
has no representative in that set. Thus it follows from Proposition 1 that the collection was not chosen to be maximal. Our claim follows.

Let \( X = \mathcal{A} \cup \bigcup_{i=1}^{n} A_{i} \). We claim that \( \pi_{1}(X) \) is the direct product of a free group with the integers. This can be seen by

1. choosing a spanning arc \( \alpha \) of \( \mathcal{A} \),
2. choosing a spanning arc \( \alpha_{i} \) of \( A_{i} \) for \( i = 1, \ldots, n \), so \( \partial \alpha_{i} = \overline{\alpha} \cap A_{i} \),
3. letting \( X_{0} = \overline{\alpha} \cup \bigcup_{i=1}^{n} \alpha_{i} \),
4. observing that \( X \) is naturally homeomorphic to \( X_{0} \times S^1 \).

Let \( N_{i} \) be a regular neighborhood of \( X \) in \( M \). Note that by construction \( N_{i} \) is homeomorphic to the product of a surface \( F \) with \( S^1 \) and that \( \lambda \) is the product of a point with \( S^1 \) in this structure. Note each component of \( \partial N_{i} \) is a torus. Suppose \( T \) is a component of \( \partial N_{i} - \mathcal{A} \) such that \( \ker(\pi_{1}(T) \to \pi_{1}(M)) \neq 1 \). Then since \( M \) is irreducible, it is a consequence of the loop theorem [4], [6] that \( T \) bounds a solid torus in \( M \). Let \( N \) be the union of \( N_{i} \) with all such solid tori. Now \( N \) is a Seifert fibre space and \( \ker(\pi_{1}(N) \to \pi_{1}(M)) = 1 \). This completes the proof of Theorem 1.

**Theorem 2.** Let \( M \) be a compact, irreducible, orientable 3-manifold with incompressible boundary and \( \lambda \) a simple loop in \( \partial M \) such that \( [\lambda] \neq 1 \in \pi_{1}(M) \). Then if there exist integers \( n > k > 1 \) and \( x \in \pi_{1}(M) \) such that \( x^{n} = [\lambda]^{k} \), there is a solid torus \( N \) embedded in \( M \) such that \( N \cap \partial M \) is an annular neighborhood of \( \lambda \) and \( \sigma \in \pi_{1}(N) \) where \( \sigma^{n} = [\lambda]^{k} \).

**Proof.** It is a consequence of the loop theorem [4], [6] that \( [\lambda] \) is of infinite order. Let \( M^{*} \) be the Seifert fibre space whose existence is guaranteed by Theorem 1. We suppose that \( M^{*} \) is the union of \( F \times S^1 \) with a collection of solid tori (regular neighborhoods of the exceptional fibres) \( N_{1}, \ldots, N_{k} \) in the interior of \( M^{*} \), that there is a point \( y \in F \) such that \( \lambda = \{y\} \times S^1 \), and \( N_{i} \cap F \times S^1 = \partial N_{i} \) for \( i = 1, \ldots, k \). Let \( \alpha_{i} \) be a simple arc properly embedded in \( F \) such that \( \partial (\alpha_{i} \times S^1) \) is the union of \( \lambda \) and a simple loop in \( \partial N_{i} \) for \( i = 1, \ldots, k \). Let \( R_{1} \) be a regular neighborhood of \( \alpha_{i} \times S^1 \cup N_{i} \) and let \( M_{1} \) be the closure of \( M^{*} - R_{1} \). Note that \( R_{1} \) is a solid torus.

By van Kampen's theorem \( \pi_{1}(M^{*}) \) is the free product of \( \pi_{1}(M_{1}) \) and \( \pi_{1}(R_{1}) \) with amalgamation over \( \pi_{1}(M_{1} \cap R_{1}) = \langle [\lambda] \rangle \). Note that any conjugate of \( x \) in \( \pi_{1}(M^{*}) \) is an \( n \)th root of \( [\lambda]^{k} \) since \( [\lambda] \) commutes with all elements of \( \pi_{1}(M^{*}) \). Now it is a consequence of Lemma 10 in [5] that we may suppose that \( x \) has length 1 since length\( [\lambda]^{k} = 1 \) and length\( [\lambda]^{k} = \text{length } x^{n} = n \cdot (\text{length } x) \) if \( (\text{length } x) > 1 \) where \( x \) is assumed to be an element represented by a cyclically reduced word. Thus we may suppose that \( x \in \pi_{1}(R_{1}) \) or \( x \in \pi_{1}(M_{1}) \).

If \( x \in \pi_{1}(R_{1}) \), we are finished. Otherwise we let \( M_{1}^{*} \) be the closure of \( M^{*} - N_{1} \). Now \( x^{n} = [\lambda]^{k} \) holds in \( \pi_{1}(M_{1}^{*}) \) since it holds in \( \pi_{1}(M_{1}) \). Furthermore \( M_{1}^{*} \) has one less singular fibre than \( M^{*} \). Since the relation \( x^{n} = [\lambda]^{k} \) holds in \( \pi_{1}(F \times S^1) \) only if \( n \) divides \( k \), the desired result follows by induction on the number of singular fibres.

In [5] Shalen points out that he has left open the following question: Does every encrusted singular curve have a crust homeomorphic to \( \mathbb{R}^{2} \times S^1 ? \)

Since an “encrusted curve” is special and any “special conjugacy class” can be represented by a power of a simple loop by Proposition 2 in [5], one is
easily able to answer the above question in the affirmative using Theorem 2 above. Jaco has also given an answer to this question in [2].

REFERENCES

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