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# THE GENUS OF AN ABSTRACT INTERSECTION SEQUENCE

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ABSTRACT. An intersection sequence, denoted *IS*, is a combinatorial object associated with a normal immersion $f\colon S^1 \to M$ where $S^1$ is an oriented circle and $M$ is a closed, connected, oriented 2-manifold. The *genus of IS*, denoted $\gamma(IS)$, is defined to be the smallest number which is the genus of a manifold $M$ admitting a realization $g\colon S^1 \to M$ of *IS*. A method is given for computing $\gamma(IS)$ from *IS*.

Throughout $S^1$ denotes a smooth circle oriented by a nonvanishing vector field $\theta$ and $M$ denotes a smooth, closed, connected, orientable 2-manifold oriented by a nonvanishing smooth 2-form $\omega$. All maps between manifolds are $C^1$. An immersion $f\colon S^1 \to M$ is called a *normal immersion* if

(a) $f^{-1}(y)$ contains at most 2 points for every $y \in M$, and

(b) $f^{-1}(y) = \{x_1, x_2\}$ implies $f'(x_1)$ and $f'(x_2)$ are linearly independent (where $f'(x) = f_*(\theta(x))$ and $f_*$ is the differential map of $f$).

It follows easily from the definition that a normal immersion has only finitely many double points, i.e. points $y \in M$ such that $f^{-1}(y)$ contains 2 points.

An *abstract intersection sequence IS* is a triple $\{n, \sigma, \nu\}$ consisting of

(a) a positive integer $n$,

(b) a bijection $\sigma\colon I_n \to I_n$, where $I_n = \{\pm 1, \ldots, \pm n\}$, such that for all $i, j \in I_n$ there is a positive integer $k$ such that $\sigma^k(i) = j$ (where $\sigma^k$ is composition of $\sigma$ with itself $k$ times) –this is just a cyclic ordering of $I_n$, and

(c) a map $\nu\colon \{1, \ldots, n\} \to \{\pm 1\}$.

If $f$ is a normal immersion with $n$ double points a *labeling* $L$ of $f$ is a naming as $y_1, \ldots, y_n$ of the double points in $M$ and a naming of their preimages

$$f^{-1}(y_i) = \{x_{-i}, x_i\}, \qquad i = 1, \ldots, n.$$

Given a labeling $L$ of the normal immersion $f$, the *intersection sequence* $IS(f, L)$ is the abstract intersection sequence $\{n, \sigma, \nu\}$ determined as follows:

(a) $n$ is the number of double points of $f$,

(b) for each $j \in I_n$, $\sigma(j)$ is the $k \in I_n$ such that $x_k$ is the first $x_i$ encountered when moving away from $x_j$ in the positive direction on $S^1$, and

(c) $\nu(j)$ is the orientation of the crossing at $y_j$ defined by

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$$\nu(j) = \operatorname{sgn}(\omega(f'(x_{-j}), f'(x_j))).$$

A normal immersion $g: S^1 \to M$ is called a *realization* of an abstract intersection sequence $IS$ if there is a labeling $L$ of $g$ such that $IS(g, L) = IS$.

Note that our definition of intersection sequences differs slightly, but not essentially, from earlier definitions. There is no need for a base point or starting point for $S^1$ when the target is a closed 2-manifold rather than the plane. Also, this definition eliminates the usual redundancy in the signing function $\nu$. See Francis [1] for a list of references and a short historical review of intersection sequences and of the problem of finding conditions on $IS$ which are necessary and sufficient for $\gamma(IS)$ to be 0. Also see Marx [3]. The problem of computing $\gamma(IS)$ generally, which we will now solve, was posed by Francis [1].

Given an abstract intersection sequence $IS = \{n, \sigma, \nu\}$ we construct a space $T(IS)$, our main tool, called the *abstract tubular neighborhood associated with* $IS$ (compare with Francis [2]) as follows: Provide $S^1$ with a Riemannian metric so that the total length of $S^1$ is $2n$. Pick any point in $S^1$, label it $x_1$ and then move in the positive direction from $x_1$ labeling the point a distance $k - 1$ from $x_1$ with the name $x_{\sigma^k(1)}$. A set of $2n$ equally spaced points around $S^1$ is obtained and these points are labeled with the names $x_{\pm 1}, \ldots, x_{\pm n}$ and are in the cyclic order determined by $\sigma$. Let $e > 0$ be a real number small enough so that the closed arcs $[x_i - e, x_i + e]$ of $S^1$ (an abuse of notation with the obvious meaning) are pairwise disjoint for $i \in I_n$. Consider the space $[-e, e] \times S^1$. It is a smooth 2-manifold with boundary which can be oriented by letting the ordered pair of tangent vectors $(d/dt, \theta)$, where $t$ is the coordinate for the interval of reals $[-e, e]$, determine the positive orientation at each point. Let $T(IS)$ be the space obtained from $[-e, e] \times S^1$ by identifying the point $(t, x_{-j} + s)$ with the point $(\nu(j)s, x_j - \nu(j)t)$ for all $t, s \in [-e, e]$. The idea is that according to whether $\nu(j)$ is positive or negative we rotate the square $[-e, e] \times [x_{-j} - e, x_{-j} + e]$ clockwise or counterclockwise through $90°$ and then identify it with the square $[-e, e] \times [x_j - e, x_j + e]$. It is easily seen that $T(IS)$ is an orientable manifold whose boundary is a collection of piecewise smooth circles. Let $T$: $[-e, e] \times S^1 \to T(IS)$ be the identification map. Then $T$ is an immersion and we can orient $T(IS)$ by taking $T$ to be orientation preserving.

Let $M(IS)$ be the smooth, closed, oriented, 2-manifold obtained from $T(IS)$ by smoothing its boundary circles and capping off the smoothed circles with 2-disks. Let $b(IS)$ be the number of circles in the boundary of $T(IS)$.

THEOREM 1. *If $IS$ is an arbitrary abstract intersection sequence, then $IS$ is realizable and*

$$\gamma(IS) = \tfrac{1}{2}(n + 2 - b(IS))$$

*which is the genus of $M(IS)$.*

PROOF. By construction, $T|\{0\} \times S^1: S^1 \to M(IS)$ is a realization of $IS$.

$M(IS)$ has a cell decomposition with $n$ 0-cells, $2n$ 1-cells and $b(IS)$ 2-cells so its Euler characteristic is $b(IS) - n$. Since the genus $g$ of $M(IS)$ is related to its Euler characteristic $h$ by the formula $\chi = 2 - 2g$, it follows that $\frac{1}{2}(n + 2 - b(IS))$ is the genus of $M(IS)$.

Now, let $f\colon S^1 \to M$ be an arbitrary realization of $IS$. It is not hard to modify the Francis [2] construction of normal tubular neighborhoods to show that $f(S^1)$ has a neighborhood $U$ in $M$ which is diffeomorphic to $T(IS)$. Then the closure of $M - U$ in $M$ is an orientable compact 2-manifold whose boundary is a union of piecewise smooth circles. If this complementary 2-manifold is not a disjoint union of 2-disks, then it contains or creates handles which make the genus of $M$ larger than the genus of $M(IS)$. Thus $\gamma(IS)$ is the genus of $M(IS)$. $\square$

Implicit in the argument above is a uniqueness result for minimum genus realizations. In fact, we easily obtain a generalization of results of Treybig [4] and Verhey [5]. Two maps $f_i\colon S^1 \to M_i$, $i = 1, 2$, are said to be *equivalent* if there exist orientation preserving diffeomorphisms $\alpha$ of $S^1$ and $\beta\colon M_1 \to M_2$ such that $f_2 = \beta \circ f_1 \circ \alpha$.

THEOREM 2. *Let $f_i\colon S^1 \to M_i$, $i = 1, 2$, be two realizations of the abstract intersection sequence IS. If the genus of both $M_1$ and $M_2$ is $\gamma(IS)$, then $f_1$ and $f_2$ are equivalent.*

SKETCH OF PROOF. The diffeomorphism $\beta$ is easily constructed to take the image $f_1(S^1)$ to the image $f_2(S^1)$ by noting that these images both have tubular neighborhoods diffeomorphic to $T(IS)$ and that these diffeomorphic tubular neighborhoods have diffeomorphic complements. Then $\alpha$ is produced by noting that $f_2$ and $\beta \circ f_1$ describe the same oriented curve in $M_2$ so since they are both immersions they differ by an orientation preserving reparameterization of $S^1$. $\square$

We finish this note by showing how to compute the number $b(IS)$ which by Theorem 1 immediately gives $\gamma(IS)$. Let

$$E_n = \{(t, x_i + s) \mid t, s = \pm e \text{ and } i \in I_n\}.$$

$E_n$ is just the set of points in $[-e, e] \times S^1$ which correspond to endpoints of maximal smooth arcs in the boundary of $T(IS)$. Let an equivalence relation $\sim$ on $E_n$ be generated by the following equivalences:

(a) $(t, x_{-j} + s) \sim (\nu(j)s, x_j - \nu(j)t)$, $t, s = \pm e$, and
(b) $(t, x_k + e) \sim (t, x_{\sigma(k)} - e)$, $t = \pm e$, $k \in I_n$.

THEOREM 3. *The number $b(IS)$ is the number of equivalence classes of the equivalence relation $\sim$ on $E_n$.*

PROOF. By (a) points of $E_n$ are equivalent if they represent the same point in $T(IS)$ and by (b) two points of $E_n$ are equivalent if they correspond to the two endpoints of a single smooth arc in the boundary of $T(IS)$. It follows immediately that two points of $E_n$ are equivalent under $\sim$ iff they project to points which are on the same component of the boundary of $T(IS)$, so there is a one-to-one correspondence between equivalence classes in $E_n$ and boundary components of $T(IS)$. $\square$

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