THE GENUS OF AN ABSTRACT INTERSECTION SEQUENCE

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Abstract. An intersection sequence, denoted IS, is a combinatorial object associated with a normal immersion \( f: S^1 \to M \) where \( S^1 \) is an oriented circle and \( M \) is a closed, connected, oriented 2-manifold. The genus of IS, denoted \( \gamma(IS) \), is defined to be the smallest number which is the genus of a manifold \( M \) admitting a realization \( g: S^1 \to M \) of IS. A method is given for computing \( \gamma(IS) \) from IS.

Throughout \( S^1 \) denotes a smooth circle oriented by a nonvanishing vector field \( \theta \) and \( M \) denotes a smooth, closed, connected, orientable 2-manifold oriented by a nonvanishing smooth 2-form \( \omega \). All maps between manifolds are \( C^1 \). An immersion \( f: S^1 \to M \) is called a normal immersion if

\[(a) \quad f^{-1}(y) \text{ contains at most 2 points for every } y \in M, \text{ and}
\[(b) \quad f^{-1}(y) = \{x_1, x_2\} \text{ implies } f'(x_1) \text{ and } f'(x_2) \text{ are linearly independent}
\] (where \( f'(x) = f_\#(\theta(x)) \) and \( f_\# \) is the differential map of \( f \)).

It follows easily from the definition that a normal immersion has only finitely many double points, i.e. points \( y \in M \) such that \( f^{-1}(y) \) contains 2 points.

An abstract intersection sequence IS is a triple \( \{n, \sigma, \nu\} \) consisting of

(a) a positive integer \( n \),

(b) a bijection \( \sigma: I_n \to I_n \), where \( I_n = \{\pm 1, \ldots, \pm n\} \), such that for all \( i, j \in I_n \) there is a positive integer \( k \) such that \( \sigma^k(i) = j \) (where \( \sigma^k \) is composition of \( \sigma \) with itself \( k \) times) –this is just a cyclic ordering of \( I_n \) and

(c) a map \( \nu: \{1, \ldots, n\} \to \{\pm 1\} \).

If \( f \) is a normal immersion with \( n \) double points a labeling \( L \) of \( f \) is a naming as \( y_1, \ldots, y_n \) of the double points in \( M \) and a naming of their preimages

\[f^{-1}(y_i) = \{x_{-i}, x_i\}, \quad i = 1, \ldots, n.\]

Given a labeling \( L \) of the normal immersion \( f \), the intersection sequence \( IS(f, L) \) is the abstract intersection sequence \( \{n, \sigma, \nu\} \) determined as follows:

(a) \( n \) is the number of double points of \( f \),

(b) for each \( j \in I_n \), \( \sigma(j) \) is the \( k \in I_n \) such that \( x_k \) is the first \( x_i \) encountered when moving away from \( x_j \) in the positive direction on \( S^1 \), and

(c) \( \nu(j) \) is the orientation of the crossing at \( y_j \) defined by

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\( r(j) = \text{sgn}(\omega(f'(x_j), f'(x_j))) \).

A normal immersion \( g: S^1 \to M \) is called a \textit{realization} of an abstract intersection sequence \( IS \) if there is a labeling \( L \) of \( g \) such that \( IS(g, L) = IS \).

Note that our definition of intersection sequences differs slightly, but not essentially, from earlier definitions. There is no need for a base point or starting point for \( S^1 \) when the target is a closed 2-manifold rather than the plane. Also, this definition eliminates the usual redundancy in the signing function \( r \). See Francis [1] for a list of references and a short historical review of intersection sequences and of the problem of finding conditions on \( IS \) which are necessary and sufficient for \( \gamma(IS) \) to be 0. Also see Marx [3]. The problem of computing \( \gamma(IS) \) generally, which we will now solve, was posed by Francis [1].

Given an abstract intersection sequence \( IS = (n, \sigma, r) \) we construct a space \( T(IS) \), our main tool, called the \textit{abstract tubular neighborhood associated with} \( IS \) (compare with Francis [2]) as follows: Provide \( S^1 \) with a Riemannian metric so that the total length of \( S^1 \) is \( 2n \). Pick any point in \( S^1 \), label it \( x_1 \) and then move in the positive direction from \( x_1 \) labeling the point a distance \( k - 1 \) from \( x_1 \) with the name \( x_{\sigma^k(1)} \). A set of \( 2n \) equally spaced points around \( S^1 \) is obtained and these points are labeled with the names \( x_{\pm 1}, \ldots, x_{\pm n} \) and are in the cyclic order determined by \( \sigma \). Let \( e > 0 \) be a real number small enough so that the closed arcs \([x_i - e, x_i + e]\) of \( S^1 \) (an abuse of notation with the obvious meaning) are pairwise disjoint for \( i \in I_n \). Consider the space \([-e, e] \times S^1 \). It is a smooth 2-manifold with boundary which can be oriented by letting the ordered pair of tangent vectors \((d/dt, \theta)\), where \( t \) is the coordinate for the interval of reals \([-e, e]\), determine the positive orientation at each point. Let \( T(IS) \) be the space obtained from \([-e, e] \times S^1 \) by identifying the point \((t, x_j + s)\) with the point \((r(j)s, x_j - r(j)t)\) for all \( t, s \in [-e, e] \). The idea is that according to whether \( r(j) \) is positive or negative we rotate the square \([-e, e] \times [x_j - e, x_j + e]\) clockwise or counterclockwise through 90° and then identify it with the square \([-e, e] \times [x_j - e, x_j + e]\). It is easily seen that \( T(IS) \) is an orientable manifold whose boundary is a collection of piecewise smooth circles. Let \( T: [-e, e] \times S^1 \to T(IS) \) be the identification map. Then \( T \) is an immersion and we can orient \( T(IS) \) by taking \( T \) to be orientation preserving.

Let \( M(IS) \) be the smooth, closed, oriented, 2-manifold obtained from \( T(IS) \) by smoothing its boundary circles and capping off the smoothed circles with 2-disks. Let \( b(IS) \) be the number of circles in the boundary of \( T(IS) \).

**Theorem 1.** If \( IS \) is an arbitrary abstract intersection sequence, then \( IS \) is realizable and

\[ \gamma(IS) = \frac{1}{2}(n + 2 - b(IS)) \]

which is the genus of \( M(IS) \).

**Proof.** By construction, \( T|\{0\} \times S^1: S^1 \to M(IS) \) is a realization of \( IS \). \( M(IS) \) has a cell decomposition with \( n \) 0-cells, \( 2n \) 1-cells and \( b(IS) \) 2-cells so its Euler characteristic is \( b(IS) - n \). Since the genus \( g \) of \( M(IS) \) is related
to its Euler characteristic \( h \) by the formula \( \chi = 2 - 2g \), it follows that \( \frac{1}{2}(n - 2 - b(IS)) \) is the genus of \( M(IS) \).

Now, let \( f: S^1 \to M \) be an arbitrary realization of \( IS \). It is not hard to modify the Francis \([2]\) construction of normal tubular neighborhoods to show that \( f(S^1) \) has a neighborhood \( U \) in \( M \) which is diffeomorphic to \( T(IS) \). Then the closure of \( M - U \) in \( M \) is an orientable compact 2-manifold whose boundary is a union of piecewise smooth circles. If this complementary 2-manifold is not a disjoint union of 2-disks, then it contains or creates handles which make the genus of \( M \) larger than the genus of \( M(IS) \). Thus \( \gamma(IS) \) is the genus of \( M(IS) \).

Implicit in the argument above is a uniqueness result for minimum genus realizations. In fact, we easily obtain a generalization of results of Treybig \([4]\) and Verhey \([5]\). Two maps \( f_i: S^1 \to M_i, i = 1, 2, \) are said to be equivalent if there exist orientation preserving diffeomorphisms \( \alpha \) of \( S^1 \) and \( \beta: M_1 \to M_2 \) such that \( f_2 = \beta \circ f_1 \circ \alpha \).

**Theorem 2.** Let \( f_i: S^1 \to M_i, i = 1, 2, \) be two realizations of the abstract intersection sequence \( IS \). If the genus of both \( M_1 \) and \( M_2 \) is \( \gamma(IS) \), then \( f_1 \) and \( f_2 \) are equivalent.

**Sketch of proof.** The diffeomorphism \( \beta \) is easily constructed to take the image \( f_1(S^1) \) to the image \( f_2(S^1) \) by noting that these images both have tubular neighborhoods diffeomorphic to \( T(IS) \) and that these diffeomorphic tubular neighborhoods have diffeomorphic complements. Then \( \alpha \) is produced by noting that \( f_2 \) and \( \beta \circ f_1 \) describe the same oriented curve in \( M_2 \) so since they are both immersions they differ by an orientation preserving reparameterization of \( S^1 \).

We finish this note by showing how to compute the number \( b(IS) \) which by Theorem 1 immediately gives \( \gamma(IS) \). Let

\[ E_n = \{(t, x_i + s) | t, s = \pm e \text{ and } i \in I_n\}. \]

\( E_n \) is just the set of points in \([-e, e] \times S^1 \) which correspond to endpoints of maximal smooth arcs in the boundary of \( T(IS) \). Let an equivalence relation \( \sim \) on \( E_n \) be generated by the following equivalences:

(a) \( (t, x_{-j} + s) \sim (v(j)s, x_j - v(j)t), t, s = \pm e \), and

(b) \( (t, x_k + e) \sim (t, x_{\sigma(k)} - e), t = \pm e, k \in I_n \).

**Theorem 3.** The number \( b(IS) \) is the number of equivalence classes of the equivalence relation \( \sim \) on \( E_n \).

**Proof.** By (a) points of \( E_n \) are equivalent if they represent the same point in \( T(IS) \) and by (b) two points of \( E_n \) are equivalent if they correspond to the two endpoints of a single smooth arc in the boundary of \( T(IS) \). It follows immediately that two points of \( E_n \) are equivalent under \( \sim \) iff they project to points which are on the same component of the boundary of \( T(IS) \), so there is a one-to-one correspondence between equivalence classes in \( E_n \) and boundary components of \( T(IS) \).

**References**

1. G. K. Francis, Null genus realizability criterion for abstract intersection sequences, J.


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