ABSOLUTELY STABLE GAMES

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Abstract. Absolutely stable games, in which every monotone chain of domination reduces to direct domination, are explicitly characterized. Simple games, and n-person games in which all minimal-vital coalitions contain at least n - 1 players, are seen to satisfy the characterization.

Harsanyi [1, pp. 1477-1479] has considered games for which all von Neumann-Morgenstern solutions exhibit a strong form of internal stability. The class of such games includes all absolutely stable games, which Harsanyi defined by a property of chains of domination. It is our purpose to give an explicit characterization of these absolutely stable games.

Let v be a (0, 1)-normalized monotonic game, with player set N. A coalition T is minimal-vital if v(T) > 0 and if for every S \subseteq T, v(S) = 0. A monotone chain is a sequence of imputations and coalitions \((x_0, S_1, x_1, \ldots, S_m, x_m)\) satisfying

1. \(x_{k-1} \text{ dom}(S_k) x_k\), and
2. \((x_0)_i > (x_k)_i\) for all \(i \in S_k\),

for \(1 \leq k \leq m\). The game is absolutely stable if for every monotone chain \((x_0, S_1, x_1, \ldots, S_m, x_m)\), it follows that \(x_0\) dominates \(x_m\).

Theorem. A game v is absolutely stable if and only if

(a) for every minimal-vital coalition T, if \(N \not\subseteq S \subseteq T\) then \(v(S) = v(T)\), and
(b) for every minimal-vital coalition T with \(v(T) < 1\), every other coalition S with \(v(S) > 0\) satisfies either \(S \supseteq T\) or \(S \supseteq N - T\).

Proof. Sufficiency. Assume that the conditions are satisfied, and consider a monotone chain \((x_0, S_1, x_1, \ldots, S_m, x_m)\). We will show that \(x_0 \text{ dom}(S_m) x_m\). By (a), all the coalitions of the chain can be assumed to be minimal-vital. By (2), \((x_0)_i > (x_m)_i\) for all \(i \in S_m\). To begin, notice that \(x_{m-1}(S_m) < v(S_m)\), where for notational convenience we generally write \(x(S) = \sum_{i \in S} x_i\). Proceeding by induction, we assume \(x_k(S_m) < v(S_m)\) and consider \(x_{k-1}\). If \(v(S_m) = 1\), then

\[x_{k-1}(S_m) \leq x_{k-1}(N) = 1 = v(S_m)\]

Otherwise (b) applies to \(S_m\), and either (i) \(S_k \supseteq S_m\), or (ii) \(S_k \supseteq N - S_m\).

If (i) applies, then since \(S_k\) is minimal-vital, it follows that \(S_k = S_m\) and therefore

\[x_{k-1}(S_m) = x_{k-1}(S_k) \leq v(S_k) = v(S_m)\]

Received by the editors January 30, 1975.

Key words and phrases. n-person games, von Neumann-Morgenstern solutions.
\[ x_{k-1}(S_m) = x_{k-1}(S_k) \leq v(S_k) = v(S_m). \]

On the other hand, if (ii) applies, then \( x_{k-1}(N - S_m) > x_k(N - S_m), \) and therefore
\[ x_{k-1}(S_m) < x_k(S_m) \leq v(S_m), \]
the last inequality following from the induction hypothesis. Thus in any case it eventually follows that \( x_0 \text{ dom}(S_m) x_m, \) and therefore \( x_0 \text{ dom}(S_m) x_m. \)

**Necessity.** Assume that \( T \) is minimal-vital with \( v(T) < 1 \), and consider any \( S \) for which \( v(S) > 0, S \supset T, \) and \( S \not\subset N - T. \) Then \( R = N - (S \cup T) \neq \emptyset \) and \( T - S \neq \emptyset. \) We write \( n, r, s, t \) for the respective cardinalities of \( N, R, S, T. \) Let
\[
\begin{align*}
x_i &= \begin{cases} 
(1 - (2s + r)e)/(t - s) & \text{if } i \in T - S, \\
2e & \text{if } i \in S, \\
e & \text{if } i \in R, 
\end{cases} \\
y_i &= \begin{cases} 
\epsilon & \text{if } i \in S \cup T, \\
(1 - (n - r)e)/r & \text{if } i \in R, 
\end{cases} \\
z_i &= \begin{cases} 
0 & \text{if } i \in T, \\
1/(n - t) & \text{if } i \in N - T. 
\end{cases}
\end{align*}
\]

Then for sufficiently small \( \epsilon > 0, x \text{ dom}(S) y \text{ dom}(T) z. \) However, \( x \) does not dominate \( z, \) since \( x(T) > v(T). \) Hence \( v \) is absolutely stable only if for every minimal-vital \( T, \) either \( v(T) = 1, \) or every \( S \) with \( v(S) > 0 \) satisfies either \( S \supset T \) or \( S \supset N - T. \) This establishes the necessity of (b).

Next, assume that \( T \) is minimal-vital with \( v(T) < 1, \) and assume that (a) fails. Then there is a minimal nonempty coalition \( S \) for which \( S \cap T = \emptyset, \ S \cup T \neq N, \) and \( v(S \cup T) > v(T). \) Select any \( k \in R = N - (S \cup T), \) and let
\[
\begin{align*}
x_i &= \begin{cases} 
(v(T) + \epsilon)/t & \text{if } i \in T, \\
\epsilon & \text{if } i \in S, \\
1 - v(T) - (s + 1)\epsilon & \text{if } i = k, \\
0 & \text{if } i \in R, i \neq k, 
\end{cases} \\
y_i &= \begin{cases} 
\epsilon & \text{if } i \in T, \\
0 & \text{if } i \in S, \\
(1 - te) & \text{if } i \in R, 
\end{cases} \\
z_i &= \begin{cases} 
0 & \text{if } i \in T, \\
1/(n - t) & \text{if } i \in N - T. 
\end{cases}
\end{align*}
\]

Then for sufficiently small \( \epsilon > 0, x \text{ dom}(S \cup T) y \text{ dom}(T) z. \) However, \( x \text{ dom}(W) z \) for all small \( \epsilon \) only if \( W = T \cup \{k\} \) and \( v(W) = 1. \) In this case, let
\[
\begin{align*}
x_i &= \begin{cases} 
(v(T) + \epsilon)/t & \text{if } i \in T, \\
1 - v(T) - \epsilon & \text{if } i = k, \\
0 & \text{if } i \in N - W. 
\end{cases}
\end{align*}
\]

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\[ y_i = \begin{cases} 
\epsilon & \text{if } i \in T, \\
0 & \text{if } i = k, \\
(1 - \epsilon)/(n - t - 1) & \text{if } i \in N - W, 
\end{cases} \]

\[ z_i = \begin{cases} 
0 & \text{if } i \neq k, \\
1 & \text{if } i = k. 
\end{cases} \]

Then for sufficiently small \( \epsilon > 0 \), \( x \) dom(\( W \)) \( y \) dom(\( T \)) \( z \), but \( x \) does not dominate \( z \). Hence \( v \) is absolutely stable only if (a) holds. This completes the proof of the theorem.

**Remark.** All simple games (games in which each \( v(S) \) is either 0 or 1) satisfy (a) because of monotonicity, and trivially satisfy (b). Games in which all minimal-vital coalitions have at least \( n - 1 \) players trivially satisfy (a), and are easily seen to satisfy (b). Games of these types were first shown to be absolutely stable in [1]. As an example of a game in neither class, \( v \) defined by

\[
\begin{align*}
&v(\{1, 2\}) = v(\{1, 2, 3\}) = v(\{1, 2, 4\}) = 1/2, \\
v(\{1, 3, 4\}) = 1/4, &v(\{2, 3, 4\}) = 4/5, &v(\{1, 2, 3, 4\}) = 1,
\end{align*}
\]

and

\( v(S) = 0 \) otherwise,

is an absolutely stable four-person game.

**Bibliography**


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