UNDECIDABILITY OF THE THEORY OF
ABELIAN
GROUPS WITH A SUBGROUP

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Abstract. The theory of abelian groups with an additional predicate
denoting a subgroup is undecidable.

0. Introduction. Let $L$ be the first order language with nonlogical symbols 0, +, $P$, where $P$ is a unary predicate symbol. For any class $K$ of abelian groups let $T(K)$ denote the $L$-theory of the class of structures $\langle A, B \rangle$, where $A \in K$ and $B \subseteq A$ is an arbitrary subgroup. Kozlov and Kokorin [4] showed that $T(K)$ is decidable if $K$ is the class of torsion free groups. The main result of this paper is the following:

Theorem 1. Let $p$ be a prime number and let $K$ be the class of abelian groups $A$ such that $p^{q}A = 0$. Then $T(K)$ is undecidable.

An immediate consequence is

Corollary 2. $T(\text{class of all abelian groups})$ is undecidable.

Corollary 2 answers a few questions asked in [4].

1. Proof of Theorem 1. Let $S$ be a finitely presented semigroup on two generators $a_1, a_2$ and defining relations $V_{\nu}(a_1, a_2) = W_{\nu}(a_1, a_2) (\nu < n)$ such that $S$ has undecidable word problem (see e.g. Davis [2]). We are going to define a finite extension $T^*$ of $T(K)$ and an effective map associating with every pair $\langle V, W \rangle$ of words in $a_1, a_2$ an $L$-sentence $\varphi$ such that $V = W$ holds in $S$ if and only if $T^* \vdash \varphi$.

Let $A$ be a $p$-group and $a \in A$. Put $\tau(a) = \langle h(a), h(pa), h(p^2a) \rangle$ where $h$ is the $p$-height, i.e. $h(x) = k$ if and only if $x \in p^kA - p^{k+1}A$. Put $\tau_0 = \langle 0, 2, 7 \rangle$, $\tau_1 = \langle 0, 3, 6 \rangle$, $\tau_2 = \langle 0, 4, 5 \rangle$. Note that for each pair $\langle i, j \rangle$, $i, j \leq 2$, $i \neq j$, $\tau_i$ has a component which is greater than the corresponding component of $\tau_j$.

For $j = 1, 2$ let $\varphi_j(x, y)$ be the $L$-formula

$$x = y = 0 \lor \exists x', y'(P(x') \& P(y') \& x = p^3x' \& y = p^3y'$$

$$\& \tau(x') = \tau_0 \& \tau(y') = \tau_j \& h(x' - y') = 1).$$

Let $T^*$ be the theory obtained from $T(K)$ by adjoining axioms (i) and (ii) below.
(i) \( \forall x(h(x) \geq 8 \rightarrow \exists y(h(y) \geq 8 \& \varphi_j(x,y))) \), \( j = 1, 2 \).

(i) simply says that \( \varphi_j \) defines a function \( f_j \) on the set of elements of height \( \geq 8 \). Therefore every word \( W(f_1, f_2) \) in \( f_1, f_2 \) defines a function \( \overline{W(f_1, f_2)} \), and a definition of this function can easily be written down in terms of \( \varphi_1, \varphi_2 \).

(ii) \( \forall x(h(x) \geq 8 \rightarrow V(f_1, f_2)(x) = \overline{W(f_1, f_2)}(x)) \), \( \nu < n \).

From (ii) it follows immediately that

\[
T^* \vdash \forall x(h(x) \geq 8 \rightarrow V(f_1, f_2)(x) = \overline{W(f_1, f_2)}(x))
\]

whenever \( V(a_1, a_2) = W(a_1, a_2) \) holds in \( S \). Since every countable semigroup can be embedded in the semigroup of endomorphisms of a countable vector space over the field \( F \) with \( p \) elements, the converse (and the theorem) clearly follow from

Claim. For any pair \( g_1, g_2 \) of endomorphisms of a countably infinite vectorspace \( V \) over \( F \) there exists a model \( \langle A, B \rangle \) of \( T(K) \) satisfying (i) such that \( \langle V, g_1, g_2 \rangle \cong \langle p^8 A, f_1, f_2 \rangle \) where \( f_1, f_2 \) are defined by \( \varphi_1, \varphi_2 \).

Proof of Claim. Put \( M = \{1, 3, 4, 6, 7, 9\} \) and for \( i \in M \) put \( A_i = (\mathbf{Z}/p^i\mathbf{Z})^{(n)}, A = \bigoplus_{i \in M} A_i \), and let \( (a_{i,k})_{k \in \omega} \) be a basis of \( A_i \). Identify \( V \) with \( p^8 A = p^8 A_9 \) and let \( a_{8,k} \in A_9 \) such that \( g_j(p^8 a_{8,k}) = p^8 a_{9,k}, j = 1, 2, k \in \omega \). For \( k \in \omega \) put

\[
\begin{align*}
b_{0,k} &= p^5 a_{9,k} + p^2 a_{4,k} + p a_{3,k} + a_{1,k}, \\
b_{1,k} &= p^5 a_{9,k}^{(1)} + p^2 a_{4,k} + a_{1,k}, \\
b_{2,k} &= p^5 a_{9,k}^{(2)} + a_{1,k},
\end{align*}
\]

and let \( B \) be the subgroup of \( A \) generated by all the \( b_{j,k} \)'s. Note that \( \sigma(b_{j,k}) = \tau_j \). The important property of these generators is

\[
\text{if } b = \sum_{i \leq 2} \sum_{k \in \omega} r_{i,k} b_{i,k}, r_{i,k} \in \mathbf{Z}, \text{ and } \tau(b) = \tau_j, \text{ then}
\]

\[
(*) \quad p \text{ divides } r_{i,k} \text{ for all } (i, k) \text{ such that } i \neq j.
\]

Assume, e.g., \( \tau(b) = \tau_1 \). Then \( p \) divides \( r_{0,k} \) because otherwise \( h(pb) = 2 \), and \( p \) divides \( r_{2,k} \) because otherwise \( h(p^2 b) = 5 \). The remaining two cases are similar.

Next we show that \( \varphi_j(a, g_j(a)) \) holds in \( \langle A, B \rangle \) for all \( a \in p^8 A, j = 1, 2 \).

This is clear if \( a = 0 \). Assume \( a = \sum_k r_k p^8 a_{9,k} \neq 0, 0 \leq r_k < p \). Put \( x' = \sum_k r_k b_{0,k}, y' = \sum_k r_k b_{j,k} \) and look at the definition of \( \varphi_j \) and \( B \).

It remains to show that \( \langle A, B \rangle \) satisfies (i). Assume \( \varphi_1(a, a_1) \) and \( \varphi_1(a, a_2) \) both hold in \( \langle A, B \rangle \) (the case \( j = 2 \) is analogous). To show: \( a_1 = a_2 \). Leaving the simpler case \( a = 0 \) to the reader we assume \( a \neq 0 \). By the definition of \( \varphi_1(x, y) \) there exist \( b_1, b_2, b_1, b_2 \in B \) such that for \( j = 1, 2 \)

\[
\begin{align*}
(1) \quad a &= p^3 b_j, \quad a_j = p^3 b_j, \\
(2) \quad \tau(b_j) &= \tau_0, \quad (\nu_j) = \tau_1, \\
(3) \quad h(b_j - b_j) &= 1.
\end{align*}
\]

Write \( b_j = \sum_{i \leq 2} \sum_{k \in \omega} s_{i,k}^{(j)} b_{i,k}, b_j = \sum_{i \leq 2} \sum_{k \in \omega} s_{i,k}^{(j)} b_{i,k}, r_{i,k}, s_{i,k} \in \mathbf{Z}. (*) \) and (2) imply
(4) \( p \) divides \( r_{1,k}^{(j)}, r_{2,k}^{(j)}, s_{0,k}^{(j)}, s_{2,k}^{(j)} \) for all \( k \in \omega, j = 1, 2 \).

This together with (3) gives

(5) \( r_{0,k}^{(j)} = s_{1,k}^{(j)} \pmod{p} \) for all \( k \in \omega, j = 1, 2 \).

(1) and (4) imply

\[
a = \sum_k r_{0,k}^{(j)} p^3 b_{0,k}, \quad a_j = \sum_k s_{1,k}^{(j)} p^3 b_{1,k}, \quad j = 1, 2.
\]

Combining the last two equations with (5) we obtain \( s_{1,k}^{(j)} = s_{2,k}^{(j)} \pmod{p} \) for all \( k \), and therefore \( a_1 = a_2 \). This proves Theorem 1.

**Remark.** Although \( T(K) \) is undecidable it is impossible to interpret number theory in it. This is a consequence of the fact that \( T(K) \) is stable in the sense of Shelah [5] whereas number theory is unstable. Stability of \( T(K) \) follows from [1] and the proof of Corollary 3 below.

### 2. Theories of modules

For any recursive ring \( R \) with identity let \( T_R \) denote the first order theory of \( R \)-modules in the language with nonlogical symbols 0, +, \( f \) (\( r \in R \)) (cf. Eklof-Sabbagh [3]). Theorem 1 can be used to prove undecidability of \( T_R \) for various rings \( R \).

**Corollary 3.** There exist finite commutative rings \( R \) such that \( T_R \) is undecidable.

**Proof.** Put \( R = R'[X]/(X^2) \) where \( R' \) is the prime ring of characteristic 29. If \( M \) is an \( R \)-module then the pair \( \mathcal{A}_M = \langle \{m \in M | Xm = 0\}, XM \rangle \), considered as a pair of abelian groups, is a model of \( T(K) \). Conversely assume \( \langle A, B \rangle \models T(K) \). Let \( B_1 \) be an isomorphic copy of \( B \) and define an endomorphism \( X \) of \( M = A \oplus B_1 \) by \( Xa = 0 \) for \( a \in A \), \( Xb_1 = b \) for \( b_1 \in B_1 \). Clearly this provides \( M \) with an \( R \)-module structure, and \( \mathcal{A}_M = \langle A, B \rangle \). This gives a faithful interpretation of \( T(K) \) in \( T_R \), hence \( T_R \) is undecidable.

Let \( F \) be a finite field. The decidability proof for the theory of abelian groups given by Szmielew [6] applies also to \( T_{F[X]} \). In contrast the next corollary shows that \( T_{F[X,Y]} \) is undecidable.

**Corollary 4.** If \( R \neq 0 \) is any recursive commutative ring then \( T_{R[X,Y]} \) is undecidable.

**Proof.** Replacing \( p \) by \( X \) and making a few obvious changes in the proof of Theorem 1 we obtain that the theory of the class of structures \( \langle M, N \rangle \), \( M \) an \( R[X] \)-module and \( N \subseteq M \) a submodule, is undecidable. By an argument similar to the one used in the proof of Corollary 3 it follows that \( T_{R[X,Y]} \) is undecidable.

**References**

4. G. T. Kozlov and A. I. Kokorin, Elementary theory of abelian groups without torsion, with a


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