UNDICIABILITY OF THE THEORY OF
ABELIAN
GROUPS WITH A SUBGROUP

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Abstract. The theory of abelian groups with an additional predicate
denoting a subgroup is undecidable.

0. Introduction. Let $L$ be the first order language with nonlogical symbols $0,$
$+,$ $,$ where $P$ is a unary predicate symbol. For any class $K$ of abelian groups
let $T(K)$ denote the $L$-theory of the class of structures $\langle A, B \rangle,$ where $A \in K$
and $B \subseteq A$ is an arbitrary subgroup. Kozlov and Kokorin [4] showed that
$T(K)$ is decidable if $K$ is the class of torsion free groups. The main result of
this paper is the following:

Theorem 1. Let $p$ be a prime number and let $K$ be the class of abelian groups
$A$ such that $p^9 A = 0.$ Then $T(K)$ is undecidable.

An immediate consequence is

Corollary 2. $T(\text{class of all abelian groups})$ is undecidable.

Corollary 2 answers a few questions asked in [4].

1. Proof of Theorem 1. Let $S$ be a finitely presented semigroup on two
generators $a_1, a_2$ and defining relations $V_i(a_1, a_2) = W_i(a_1, a_2)(i > n)$ such
that $S$ has undecidable word problem (see e.g. Davis [2]). We are going to
define a finite extension $T^*$ of $T(K)$ and an effective map associating with
every pair $\langle V, W \rangle$ of words in $a_1, a_2$ an $L$-sentence $\varphi$ such that $V = W$ holds
in $S$ if and only if $T^* \models \varphi.$

Let $A$ be a $p$-group and $a \in A.$ Put $t(a) = (h(a), h(p a), h(p^2 a))$ where $h$
is the $p$-height, i.e. $h(x) = k$ if and only if $x \in p^k A - p^{k+1} A.$ Put $\tau_0 = \langle 0, 2, 7 \rangle,$
$\tau_1 = \langle 0, 3, 6 \rangle,$ $\tau_2 = \langle 0, 4, 5 \rangle.$ Note that for each pair $\langle i, j \rangle,$ $i, j \leq 2,$ $i \neq j,$ $\tau_i$
has a component which is greater than the corresponding component of $\tau_j.$

For $j = 1, 2$ let $\varphi_j(x, y)$ be the $L$-formula

$x = y = 0 \lor \exists x', y'(p(x') \land P(y') \land x = p^3 x' \land y = p^3 y'$

$& \tau(x') = \tau_0 \land \tau(y') = \tau_j \land h(x' - y') = 1).$

Let $T^*$ be the theory obtained from $T(K)$ by adjoining axioms (i) and (ii)
below.

Received by the editors March 26, 1975.

AMS (MOS) subject classifications (1970). Primary 02G05.

1 Supported by Schweizerischer Nationalfonds.

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\[(i) \quad \forall x(h(x) \geq 8 \rightarrow \exists ! y(h(y) \geq 8 \& \varphi_j(x,y))), \quad j = 1, 2.\]

(i) simply says that \(\varphi_j\) defines a function \(f_j\) on the set of elements of height \(\geq 8\). Therefore every word \(W(f_1, f_2)\) in \(f_1, f_2\) defines a function \(W(f_1, f_2)\), and a definition of this function can easily be written down in terms of \(\varphi_1, \varphi_2\).

\[(ii) \quad \forall x(h(x) \geq 8 \rightarrow V_p(f_1, f_2)(x) = \overline{W_p(f_1, f_2)(x)}), \quad \nu < n.\]

From (ii) it follows immediately that

\[T^* \vdash \forall x(h(x) \geq 8 \rightarrow V_p(f_1, f_2)(x) = \overline{W_p(f_1, f_2)(x)})\]

whenever \(V(a_1, a_2) = W(a_1, a_2)\) holds in \(S\). Since every countable semigroup can be embedded in the semigroup of endomorphisms of a countable vector space over the field \(F\) with \(p\) elements, the converse (and the theorem) clearly follow from

Claim. For any pair \(g_1, g_2\) of endomorphisms of a countably infinite vectorspace \(V\) over \(F\) there exists a model \(\langle A, B \rangle\) of \(T(K)\) satisfying (i) such that \(\langle V, g_1, g_2 \rangle \cong \langle p^8 A, f_1, f_2 \rangle\) where \(f_1, f_2\) are defined by \(\varphi_1, \varphi_2\).

Proof of Claim. Put \(M = \{1, 3, 4, 6, 7, 9\}\) and for \(i \in M\) put \(A_i = (\mathbb{Z}/p^i\mathbb{Z})^\omega, A = \bigoplus_{i \in M} A_i\), and let \((a_{i,k})_{k \in \omega}\) be a basis of \(A_i\). Identify \(V\) with \(p^8 A = p^8 A_9\) and let \(a_{9,j,k} \in A_9\) such that \(g_j(p^8 a_{9,k}) = p^8 a_{9,j,k}\), \(j = 1, 2, k \in \omega\). For \(k \in \omega\) put

\[
\begin{align*}
  b_{0,k} &= p^5 a_{9,k} + pa_{3,k} + a_{1,k} \\
  b_{1,k} &= p^5 a_{9,k}^{(1)} + p^4 a_{7,k} + p^2 a_{4,k} + a_{1,k} \\
  b_{2,k} &= p^5 a_{9,k}^{(2)} + p^3 a_{6,k} + a_{1,k}
\end{align*}
\]

and let \(B\) be the subgroup of \(A\) generated by all the \(b_{j,k}\)'s. Note that \(\tau(b_{j,k}) = \tau_j\). The important property of these generators is

\[(*) \quad p \text{ divides } r_{i,k} \text{ for all } \langle i, k \rangle \text{ such that } i \neq j.\]

Assume, e.g., \(\tau(b) = \tau_1\). Then \(p\) divides \(r_{0,k}\) because otherwise \(h(p b) = 2\), and \(p\) divides \(r_{2,k}\) because otherwise \(h(p^2 b) = 5\). The remaining two cases are similar.

Next we show that \(\varphi_j(a, g_j(a))\) holds in \(\langle A, B \rangle\) for all \(a \in p^8 A, j = 1, 2\). This is clear if \(a = 0\). Assume \(a = \sum_k r_k p^8 a_{9,k} \neq 0, 0 \leq r_k < p\). Put \(x' = \sum_k r_k b_{0,k}, y' = \sum_k r_k b_{j,k}\) and look at the definition of \(g_j\) and \(B\).

It remains to show that \(\langle A, B \rangle\) satisfies (i). Assume \(\varphi_1(a, a_1)\) and \(\varphi_1(a, a_2)\) both hold in \(\langle A, B \rangle\) (the case \(j = 2\) is analogous). To show: \(a_1 = a_2\). Leaving the simpler case \(a = 0\) to the reader we assume \(a \neq 0\). By the definition of \(\varphi_1(x, y)\) there exist \(b_1', b_2', b_1, b_2 \in B\) such that for \(j = 1, 2\)

1. \(a = p^3 b_j', a_j = p^3 b_j',\)
2. \(\tau(b_j') = \tau_0, \tau(b_j) = \tau_1,\)
3. \(h(b_j' - b_j) = 1,\)

Write \(b_j = \sum_{i \leq 2} \sum_{k \in \omega} r_{i,k}^{(j)} b_{i,k}, b_j = \sum_{i \leq 2} \sum_{k \in \omega} s_{i,k}^{(j)} b_{i,k}, r_{i,k}^{(j)}, s_{i,k}^{(j)} \in \mathbb{Z}\). (*) and (2) imply
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(4) $p$ divides $r_{0,k}^{(j)}$, $r_{2,k}^{(j)}$, $s_{0,k}^{(j)}$, $s_{2,k}^{(j)}$ for all $k \in \omega, j = 1, 2$.

This together with (3) gives

(5) $r_{0,k}^{(j)} \equiv s_{1,k}^{(j)} (\mod p)$ for all $k \in \omega, j = 1, 2$.

(1) and (4) imply

$$a = \sum_{k} r_{0,k}^{(j)} p^3 b_{0,k}, \quad a_j = \sum_{k} s_{1,k}^{(j)} p^3 b_{1,k}, \quad j = 1, 2.$$ 

Combining the last two equations with (5) we obtain $s_{1,k}^{(j)} \equiv s_{1,k}^{(j)} (\mod p)$ for all $k$, and therefore $a_1 = a_2$. This proves Theorem 1.

**Remark.** Although $T(K)$ is undecidable it is impossible to interpret number theory in it. This is a consequence of the fact that $T(K)$ is stable in the sense of Shelah [5] whereas number theory is unstable. Stability of $T(K)$ follows from [1] and the proof of Corollary 3 below.

2. **Theories of modules.** For any recursive ring $R$ with identity let $T_R$ denote the first order theory of $R$-modules in the language with nonlogical symbols 0, +, $f_r$ ($r \in R$) (cf. Eklof-Sabbagh [3]). Theorem 1 can be used to prove undecidability of $T_R$ for various rings $R$.

**Corollary 3.** There exist finite commutative rings $R$ such that $T_R$ is undecidable.

**Proof.** Put $R = R'[X]/(X^2)$ where $R'$ is the prime ring of characteristic $2^9$. If $M$ is an $R$-module then the pair $\mathfrak{A}_M = \langle \{m \in M | Xm = 0\}, XM \rangle$, considered as a pair of abelian groups, is a model of $T(K)$. Conversely assume $\langle A, B \rangle \models T(K)$. Let $B_1$ be an isomorphic copy of $B$ and define an endomorphism $X$ of $M = A \oplus B_1$ by $Xa = 0$ for $a \in A$, $Xb_1 = b$ for $b_1 \in B_1$. Clearly this provides $M$ with an $R$-module structure, and $\mathfrak{A}_M = \langle A, B \rangle$. This gives a faithful interpretation of $T(K)$ in $T_R$, hence $T_R$ is undecidable.

Let $F$ be a finite field. The decidability proof for the theory of abelian groups given by Szmielew [6] applies also to $T_{F[X]}$. In contrast the next corollary shows that $T_{F[X,Y]}$ is undecidable.

**Corollary 4.** If $R \neq 0$ is any recursive commutative ring then $T_{R[X,Y]}$ is undecidable.

**Proof.** Replacing $p$ by $X$ and making a few obvious changes in the proof of Theorem 1 we obtain that the theory of the class of structures $\langle M, N \rangle$, $M$ an $R[X]$-module and $N \subseteq M$ a submodule, is undecidable. By an argument similar to the one used in the proof of Corollary 3 it follows that $T_{R[X,Y]}$ is undecidable.

**References**

4. G. T. Kozlov and A. I. Kokorin, Elementary theory of abelian groups without torsion, with a


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