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UNDECIDABILITY OF THE THEORY OF ABELIAN GROUPS WITH A SUBGROUP

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ABSTRACT. The theory of abelian groups with an additional predicate denoting a subgroup is undecidable.

0. **Introduction.** Let L be the first order language with nonlogical symbols 0, $+$, P, where P is a unary predicate symbol. For any class K of abelian groups let $T(K)$ denote the L-theory of the class of structures $\langle A, B\rangle$, where $A \in K$ and $B \subseteq A$ is an arbitrary subgroup. Kozlov and Kokorin [4] showed that $T(K)$ is decidable if K is the class of torsion free groups. The main result of this paper is the following:

THEOREM 1. *Let p be a prime number and let K be the class of abelian groups A such that $p^9 A = 0$. Then $T(K)$ is undecidable.*

An immediate consequence is

COROLLARY 2. *T(class of all abelian groups) is undecidable.*

Corollary 2 answers a few questions asked in [4].

1. **Proof of Theorem 1.** Let S be a finitely presented semigroup on two generators α_1, α_2 and defining relations $V_\nu(\alpha_1,\alpha_2) = W_\nu(\alpha_1,\alpha_2)(\nu < n)$ such that S has undecidable word problem (see e.g. Davis [2]). We are going to define a finite extension T^* of $T(K)$ and an effective map associating with every pair $\langle V, W\rangle$ of words in α_1, α_2 an L-sentence φ such that $V = W$ holds in S if and only if $T^* \vdash \varphi$.

Let A be a p-group and $a \in A$. Put $\tau(a) = \langle h(a), h(pa), h(p^2 a)\rangle$ where h is the p-height, i.e. $h(x) = k$ if and only if $x \in p^\kappa A - p^{\kappa+1}A$. Put $\tau_0 = \langle 0,2,7\rangle$, $\tau_1 = \langle 0,3,6\rangle$, $\tau_2 = \langle 0,4,5\rangle$. Note that for each pair $\langle i,j\rangle$, $i,j \leqslant 2$, $i \neq j$, τ_i has a component which is greater than the corresponding component of τ_j.

For $j = 1, 2$ let $\varphi_j(x,y)$ be the L-formula

$$x = y = 0 \vee \exists x', y'(P(x') \,\&\, P(y') \,\&\, x = p^3 x' \,\&\, y = p^3 y' \\ \&\, \tau(x') = \tau_0 \,\&\, \tau(y') = \tau_j \,\&\, h(x' - y') = 1).$$

Let T^* be the theory obtained from $T(K)$ by adjoining axioms (i) and (ii) below.

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(i) $\forall x(h(x) \geqslant 8 \to \exists! y(h(y) \geqslant 8 \;\&\; \varphi_j(x,y))), \qquad j = 1, 2.$

(i) simply says that φ_j defines a function f_j on the set of elements of height $\geqslant 8$. Therefore every word $W(f_1,f_2)$ in f_1, f_2 defines a function $\overline{W(f_1,f_2)}$, and a definition of this function can easily be written down in terms of φ_1, φ_2.

(ii) $\forall x(h(x) \geqslant 8 \to \overline{V_\nu(f_1,f_2)}(x) = \overline{W_\nu(f_1,f_2)}(x)), \quad \nu < n.$

From (ii) it follows immediately that

$$T^* \vdash \forall x(h(x) \geqslant 8 \to \overline{V(f_1,f_2)}(x) = \overline{W(f_1,f_2)}(x))$$

whenever $V(\alpha_1,\alpha_2) = W(\alpha_1,\alpha_2)$ holds in S. Since every countable semigroup can be embedded in the semigroup of endomorphisms of a countable vector space over the field F with p elements, the converse (and the theorem) clearly follow from

Claim. For any pair g_1, g_2 *of endomorphisms of a countably infinite vectorspace* V *over* F *there exists a model* $\langle A, B\rangle$ *of* $T(K)$ *satisfying* (i) *such that* $\langle V, g_1, g_2\rangle \cong \langle p^8 A, f_1, f_2\rangle$ *where* f_1, f_2 *are defined by* φ_1, φ_2.

PROOF OF CLAIM. Put $M = \{1,3,4,6,7,9\}$ and for $i \in M$ put $A_i = (\mathbf{Z}/p^i\mathbf{Z})^{(\omega)}$, $A = \bigoplus_{i \in M} A_i$, and let $(a_{i,k})_{k\in\omega}$ be a basis of A_i. Identify V with $p^8 A = p^8 A_9$ and let $a_{9,k}^{(j)} \in A_9$ such that $g_j(p^8 a_{9,k}) = p^8 a_{9,k}^{(j)}$, $j = 1, 2$, $k \in \omega$. For $k \in \omega$ put

$$\begin{aligned}
b_{0,k} &= p^5 a_{9,k} && && && +pa_{3,k} + a_{1,k}\\
b_{1,k} &= p^5 a_{9,k}^{(1)} && +p^4 a_{7,k} && && +p^2 a_{4,k} \quad + a_{1,k}\\
b_{2,k} &= p^5 a_{9,k}^{(2)} && && +p^3 a_{6,k} && + a_{1,k}
\end{aligned}$$

and let B be the subgroup of A generated by all the $b_{j,k}$'s. Note that $\tau(b_{j,k}) = \tau_j$. The important property of these generators is

$$(*) \qquad \begin{gathered}\text{if } b = \textstyle\sum_{i\leqslant 2}\sum_{k\in\omega} r_{i,k} b_{i,k},\ r_{i,k} \in \mathbf{Z}, \text{ and } \tau(b) = \tau_j, \text{ then}\\ p \text{ divides } r_{i,k} \text{ for all } \langle i,k\rangle \text{ such that } i \neq j.\end{gathered}$$

Assume, e.g., $\tau(b) = \tau_1$. Then p divides $r_{0,k}$ because otherwise $h(pb) = 2$, and p divides $r_{2,k}$ because otherwise $h(p^2 b) = 5$. The remaining two cases are similar.

Next we show that $\varphi_j(a, g_j(a))$ holds in $\langle A, B\rangle$ for all $a \in p^8 A$, $j = 1, 2$. This is clear if $a = 0$. Assume $a = \sum_k r_k p^8 a_{9,k} \neq 0$, $0 \leqslant r_k < p$. Put $x' = \sum_k r_k b_{0,k}$, $y' = \sum_k r_k b_{j,k}$ and look at the definition of φ_j and B.

It remains to show that $\langle A, B\rangle$ satisfies (i). Assume $\varphi_1(a, a_1)$ and $\varphi_1(a, a_2)$ both hold in $\langle A, B\rangle$ (the case $j = 2$ is analogous). To show: $a_1 = a_2$. Leaving the simpler case $a = 0$ to the reader we assume $a \neq 0$. By the definition of $\varphi_1(x, y)$ there exist b_1', b_2', b_1, $b_2 \in B$ such that for $j = 1, 2$

(1) $a = p^3 b_j'$, $a_j = p^3 b_j$,

(2) $\tau(b_j') = \tau_0$, $\tau(b_j) = \tau_1$,

(3) $h(b_j' - b_j) = 1$.

Write $b_j' = \sum_{i\leqslant 2}\sum_{k\in\omega} r_{i,k}^{(j)} b_{i,k}$, $b_j = \sum_{i\leqslant 2}\sum_{k\in\omega} s_{i,k}^{(j)} b_{i,k}$, $r_{i,k}^{(j)}$, $s_{i,k}^{(j)} \in \mathbf{Z}$. (*) and (2) imply

(4) $\qquad p$ divides $r_{1,k}^{(j)}, r_{2,k}^{(j)}, s_{0,k}^{(j)}, s_{2,k}^{(j)}$ for all $k \in \omega, j = 1, 2$.

This together with (3) gives

(5) $\qquad r_{0,k}^{(j)} \equiv s_{1,k}^{(j)} \pmod p$ for all $k \in \omega, j = 1, 2$.

(1) and (4) imply

$$a = \sum_k r_{0,k}^{(j)} p^3 b_{0,k}, \quad a_j = \sum_k s_{1,k}^{(j)} p^3 b_{1,k}, \qquad j = 1, 2.$$

Combining the last two equations with (5) we obtain $s_{1,k}^{(1)} \equiv s_{1,k}^{(2)} \pmod p$ for all k, and therefore $a_1 = a_2$. This proves Theorem 1.

REMARK. Although $T(K)$ is undecidable it is impossible to interprete number theory in it. This is a consequence of the fact that $T(K)$ is stable in the sense of Shelah [5] whereas number theory is unstable. Stability of $T(K)$ follows from [1] and the proof of Corollary 3 below.

2. **Theories of modules.** For any recursive ring R with identity let T_R denote the first order theory of R-modules in the language with nonlogical symbols 0, $+$, f_r $(r \in R)$ (cf. Eklof-Sabbagh [3]). Theorem 1 can be used to prove undecidability of T_R for various rings R.

COROLLARY 3. *There exist finite commutative rings R such that T_R is undecidable.*

PROOF. Put $R = R'[X]/(X^2)$ where R' is the prime ring of characteristic 2^9. If M is an R-module then the pair $\mathfrak{A}_M = \langle \{m \in M | Xm = 0\}, XM \rangle$, considered as a pair of abelian groups, is a model of $T(K)$. Conversely assume $\langle A, B \rangle \models T(K)$. Let B_1 be an isomorphic copy of B and define an endomorphism X of $M = A \oplus B_1$ by $Xa = 0$ for $a \in A$, $Xb_1 = b$ for $b_1 \in B_1$. Clearly this provides M with an R-module structure, and $\mathfrak{A}_M = \langle A, B \rangle$. This gives a faithful interpretation of $T(K)$ in T_R, hence T_R is undecidable.

Let F be a finite field. The decidability proof for the theory of abelian groups given by Szmielew [6] applies also to $T_{F[X]}$. In contrast the next corollary shows that $T_{F[X,Y]}$ is undecidable.

COROLLARY 4. *If $R \neq 0$ is any recursive commutative ring then $T_{R[X,Y]}$ is undecidable.*

PROOF. Replacing p by X and making a few obvious changes in the proof of Theorem 1 we obtain that the theory of the class of structures $\langle M, N \rangle$, M an $R[X]$-module and $N \subseteq M$ a submodule, is undecidable. By an argument similar to the one used in the proof of Corollary 3 it follows that $T_{R[X,Y]}$ is undecidable.

REFERENCES

1. W. Baur, *$\aleph_0$-categorical modules*, J. Symbolic Logic **40** (1975), 213–220.

2. M. Davis, *Computability and unsolvability*, McGraw-Hill Ser. in Information Processing and Computers, McGraw-Hill, New York, 1958. MR **23** #A1525.

3. P. C. Eklof and G. Sabbagh, *Model-completions and modules*, Ann. Math. Logic **2** (1970/71), no. 3, 251–295. MR **43** #3105.

4. G. T. Kozlov and A. I. Kokorin, *Elementary theory of abelian groups without torsion, with a*

predicate selecting a subgroup, Algebra i Logika **8** (1969), 320–334 = Algebra and Logic **8** (1969), 182–190. MR **41** #3263.

5. S. Shelah, *Stability, the f.c.p., and superstability: Model theoretic properties of formulas in first order theory*, Ann. Math. Logic **3** (1971), no. 3, 271–362. MR **47** #6475.

6. W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. **41** (1955), 203–271. MR **17**, 233.

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