

AN EXTENSION THEOREM FOR FUNCTIONS ON SEMIGROUPS

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ABSTRACT. For S a semitopological semigroup, a continuous function on S is said to be in $LMC(S)$ if its set of right translates is relatively compact in $C(S)$ for the topology of pointwise convergence on S . It is proved here that, if S is a dense subsemigroup of a topological group G , then every function in $LMC(S)$ extends to a function continuous on G . This result generalizes earlier results that were arrived at independently by A. T. Lau and the present author. Some corollaries of this result are also presented.

1. Definitions and main result. Let S be a topological semigroup, that is, S is a semigroup and a Hausdorff topological space, and the binary operation $(s, t) \rightarrow st$ from the product topological space $S \times S$ into S is continuous. A function $f \in C(S)$ is said to be in $AP(S)$ ($WAP(S)$) [$LMC(S)$] provided its set of right translates is relatively compact in $C(S)$ for the topology of norm convergence (weak convergence) [pointwise convergence on S], the *left* and *right translates*, f_s and f^s , of $f \in C(S)$ by $s \in S$ being defined as usual by

$$f_s(t) = f(st) \quad \text{and} \quad f^s(t) = f(ts) \quad \text{for all } t \in S,$$

respectively. $AP(S)$ and $WAP(S)$ are the spaces of *almost periodic* and *weakly almost periodic* functions in $C(S)$ and, if βS is the spectrum of $C(S)$ (which is just the Stone-Čech compactification of S if S is completely regular), then it is not hard to show that a function $f \in C(S)$ is in $LMC(S)$ if and only if the function $s \rightarrow x(f_s)$, $s \in S$, is in $C(S)$ for each $x \in \beta S$. And, finally, a function $f \in C(S)$ is called *left* (*right*) *uniformly continuous* if the function, $s \rightarrow f_s$ ($s \rightarrow f^s$), from S into $C(S)$ is norm-continuous, and *uniformly continuous* if it is both left and right uniformly continuous. Notations for these last families of functions are $LUC(S)$, $RUC(S)$ and $UC(S)$, respectively. The reader is referred to the references for more information about these function spaces.

We are now ready to state and prove our main result, which generalizes both Lemma 2.1 of [5] and Theorem 1 of [6] by achieving the same conclusion from weaker hypotheses. The proof is essentially that of [6], the difference being that a device of Lau is used to get around the possible absence of the identity in S .

THEOREM. *Let S be a dense subsemigroup of a topological semigroup G that is*

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algebraically a group. Then every $f \in LMC(S)$ extends to a function in $C(G)$, i.e., $C(G)|_S \supseteq LMC(S)$.

PROOF. If S is a dense subsemigroup of G and $r \in S$, then $T = rSr$ is a subsemigroup of S that is dense in G ; for, the map $t \rightarrow rtr$ is a homeomorphism of G onto itself, since G is a group and a topological semigroup. We now produce a contradiction from the assumption that there exist $s \in G$ and nets $\{s'_\alpha\}$ and $\{t'_\beta\}$ in T such that $\lim_\alpha s'_\alpha = s = \lim_\beta t'_\beta$ and $a = \lim_\alpha f(s'_\alpha) \neq \lim_\beta f(t'_\beta) = b$ for an $f \in LMC(S)$. This will complete the proof.

Let $s_\alpha = r^{-1}s'_\alpha$ for each α and let $t_\beta = t'_\beta r^{-1}$ for each β . Then $\{s_\alpha\} \subset S$ and $\{t_\beta\} \subset S$. Without loss we may assume $\lim_\alpha e(s_\alpha) = x \in \beta S$, where e is the canonical continuous map of S into βS ; thus $x(f_r) = a$. Since T is dense in G , there is a net $\{r_\gamma\} \subset T$ such that $\lim_\gamma r_\gamma = s^{-1} \in G$. Now consider the values f along the “triple net” $\{t_\beta rr_\gamma rs_\alpha\} \subset S$. Since $f \in LMC(S)$, the function $t \rightarrow x(f_t)$ is continuous on S and, since $t_\beta rr_\gamma r \rightarrow r \in S$ by continuity of multiplication, $x(f_{t_\beta rr_\gamma r})$ should approach $x(f_r) = a$. But

$$x(f_{t_\beta rr_\gamma r}) = \lim_\alpha f(t_\beta rr_\gamma rs_\alpha)$$

is close to $f(t_\beta r) = f(t'_\beta)$ for all large enough γ since $r_\gamma rs_\alpha \rightarrow e$. This implies that

$$x(f_{t_\beta rr_\gamma r}) \rightarrow \lim_\beta f(t_\beta r) = b \neq a = x(f_r)$$

as $t_\beta rr_\gamma r \rightarrow r$, the desired contradiction.

Counterexamples convince us that the hypotheses of the theorem cannot be weakened much. However, the conclusion may be too weak; we now state a conjecture which, to our knowledge, has been verified only in the case that G is compact (Corollary 1, ahead).

CONJECTURE. In the setting of the theorem, $LMC(G)|_S = LMC(S)$. (It follows directly from definitions that $LMC(G)|_S \subset LMC(S)$.)

2. Corollaries to the main result. We now state a number of corollaries which generalize results of [5], [6]. They can be proved using methods of these papers and the theorem above. And, once again, counterexamples show that most hypotheses cannot be weakened much. (See the Remark below.)

COROLLARY 1. If S is a dense subsemigroup of a compact topological group G , then $LMC(G)|_S = LMC(S)$. Hence $LMC(S) = AP(S)$.

COROLLARY 2. If S is a dense subsemigroup of a topological semigroup G that is algebraically a group, then $AP(G)|_S = AP(S)$ and $WAP(G)|_S = WAP(S)$. If, as well, G is locally compact or a complete metric space, then $WAP(S) \subset UC(S)$.

COROLLARY 3. If S is a subsemigroup of a topological group G and S has compact closure in G , then $LUC(G)|_S = LMC(S)$. If, as well, G is locally compact, then $WAP(G)|_S = LMC(S)$.

COROLLARY 4. If S is a subsemigroup of a locally compact abelian group G and the closure of S in G is a group, then $AP(G)|_S = AP(S)$.

COROLLARY 5. *If S is a subsemigroup of a locally compact group G and the closure of S in G is an open subgroup of G , then $\text{WAP}(G)|_S = \text{WAP}(S)$.*

REMARK. It follows from a fundamental theorem of A. Weil on the completion of topological groups [8, p. 32] that the terms “compact group” and “locally compact group” can be weakened to “totally bounded topological group” and “topological group with a totally bounded neighbourhood”, respectively, in these corollaries.

Added in proof. Dr. J. S. Pym has informed the author that the Conjecture stated above is false.

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