REMARKS ON H. H. SCHAEFER'S CONDITIONS

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Abstract. We present two algebraic conditions, each characterizing a partially ordered linear algebra as an algebra of real-valued functions.

In this note we present two algebraic conditions which characterize a Dedekind σ-complete partially ordered linear algebra (dsc-pola) as an algebra of real-valued functions. Work of this type has been done by DeMarr [2], [3]. These two conditions are motivated by Schaefer's work [5, p. 169]; see also [4, p. 2108]. He used these conditions together to characterize a so-called real spectral operator [5, Definitions 3,4] on a locally convex linear space. Roughly speaking a real spectral operator is like a real-valued function.

A dsc-pola $A$ is a real linear associative algebra which satisfies the following conditions. (1) It is partially ordered so that it is a directed partially ordered linear space and $0 \leq xy$ whenever $0 \leq x, 0 \leq y$. (2) It is Dedekind σ-complete, i.e., for $x_n \in A$ and $x_1 \geq x_2 \geq \cdots \geq 0$ implies $\inf \{x_n\}$ exists. In this paper we assume that $A$ has multiplicative identity $1 > 0$. Let $I = \{y \in A: y \geq 1, y^{-1} > 0\}$. Define $A_1 = \bigcup_{y \in I} \{x \in A: -y \leq x \leq y\}$. Then the multiplication of the elements in $A_1$ is commutative, and $A_1$ behaves much like an algebra of real-valued functions. Of particular importance is that the $A_1$ is an order closed and order convex subalgebra and an f-ring [1, p. 403]. For details of proofs see [2].

Now we consider Schaefer's two conditions as applied to a dsc-pola $A$. In each case we will show that $A = A_1$.

Theorem 1. Suppose $A$ has the following property: for any $x \in A$, $(1 + x^2)^{-1}$ exists and $(1 + x^2)^{-1} \leq 1$. Then $A = A_1$.

Proof. First we show that if $y > 1$ and $y^{-1}$ exists, then $y \in A_1$. By assumption $(1 + y^{-2})^{-1}$ exists and $y^{-2}(1 + y^{-2})^{-1} = 1 - (1 + y^{-2})^{-1} \geq 0$; therefore, $(1 + y^{-2})^{-1} \geq 0$, so $(1 + y^{-2})^{-1} \in A_1$. This implies $1 \leq 1 + y^{-2} \in A_1$ [2, Theorem I.6.10] or $y^{-2} \geq 0$. Hence, by definition of $A_1$ we know $y^2 \in A_1$. Since $1 \leq y \leq y^2$ and $A_1$ is order convex we get $y \in A_1$. Now for any $1 \leq x \in A$, clearly $1 \leq x \leq 1 + x^2$. By assumption $(1 + x^2)^{-1}$ exists and from the above paragraph we get $1 + x^2 \in A_1$. Since $A_1$ is order convex we get $x \in A_1$. Since $A$ is directed we get $A = A_1$. □

Remark. The above theorem is true if we consider the following weaker
property: for any \( x \in A \) there exists a strictly positive integer \( m \) such that \((1 + x^m)^{-1}\) exists and \((1 + x^m)^{-1} \leq 1\).

**Theorem 2.** Suppose \( A \) has the following property: for any \( x \in A \), \((1 + x^2)^{-1}\) exists and \(x(1 + x^2)^{-1} \leq 1\). Then \( A = A_1 \).

**Proof.** Replacing \( x \) by \(-x\) in the assumption we see easily that \(-1 \leq x(1 + x^2)^{-1} \leq 1\). Thus for any \( x \in A \), \((1 + x^2)^{-1} \leq A_1 \). Take any \( 1 \leq z \in A \). Clearly \( 1 \leq z \leq 1 + z^2 = y \) and \( y^{-1} \) exists. Therefore for all positive integers \( n \), \(-1 \leq (y/n)(1 + (y/n)^2)^{-1} \leq 1\). Hence \( y^{-1}(1 + (y/n)^2) = y^{-1} + y/n^2 \in A_1 \). Since \( A_1 \) is closed (order) we have \( y^{-1} = \lim_{n \to \infty} (y^{-1} + y/n^2) \in A_1 \), so that \( y \in A_1 \). This means \( z \in A_1 \). Since \( A \) is directed we have \( A = A_1 \). 

**Remarks.** (1) It can be easily shown that the conditions mentioned above are also necessary conditions. Hence the conditions are equivalent.

(2) Since \( A_1 \) itself is a Dedekind \( \sigma \)-complete lattice and since \( A_1 \) is an \( f \)-ring we can easily verify that \( 1 \) is a weak unit, i.e., for \( z \geq 0 \), \( 1 \wedge z = 0 \) imply \( z = 0 \). Thus for any \( x \in A_1 \) by Freudenthal theorem [1, p. 364] we know \( x = \int_{-\infty}^{\infty} \lambda \, d\nu(\lambda) \). Where \( \lambda \) is real, \( 1 \geq \nu(\lambda)^2 = \nu(\lambda) \geq 0 \) and the integral is defined to be \( o - \lim_{n \to \infty} \sum_{k=-n}^{n} (k/n)(\nu((k+1)/n) - \nu(k/n)) \). This means every \( x \in A_1 \) is a spectral element in the sense of Schaefer [5, Definitions 3,4].

**References**


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