COMPACTNESS OF CERTAIN HOMOGENEOUS SPACES OF LOCALLY COMPACT GROUPS

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Abstract. Let $H$ be the fixed points of a family of automorphisms of a locally compact group $G$ with $G/H$ finite invariant measure. It is proved in this paper that when the $1$-component of $G$ is open, $G/H$ is compact.

Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$ such that $G/H$ admits a finite $G$-invariant measure. Then $G/H$ is compact if $G$ is a connected Lie group and $H$ has finitely many connected components [6, Mostow], or if $G$ is a $p$-adic group and $H$ is discrete [8, Tamagawa]. Recently, Greenleaf-Moskowitz-Rothschild [1], [2] proved that $G/H$ is compact for disconnected Lie groups $G$ with $H$ consisting of the fixed points of a family of automorphisms of $G$ (see Lemma 3 below). Under similar restrictions on $H$ as in [1], the author [7] obtained the same result for linear algebraic groups defined over locally compact fields. Now in this paper, we prove the following theorem, which extends Lemma 3 to non-Lie groups.

Theorem. Let $G$ be a locally compact group and $H$ be a closed subgroup consisting of the fixed points of a family of automorphisms of $G$ such that $G/H$ has a finite $G$-invariant measure. If $G$ is $\sigma$-compact with its $1$-component open then $G/H$ is compact. In particular, this is the case when $G$ is connected or when $G$ is $\sigma$-compact and locally connected.

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1. Preliminaries and notations. Throughout this paper we consider only $\sigma$-compact groups (i.e. groups which are countable union of compact subsets). For a locally compact group $G$, let $G_0$ denote its $1$-component, $\mathcal{A}(G)$ the group of topological automorphisms of $G$ and $\mathcal{G}(G) = \{\alpha_x | x \in G\}$ the subgroup of inner automorphisms of $G$. Let $K(G_0)$ denote the maximal compact normal subgroup of $G_0$ (the existence of such a group is proved in [4]). For a subset $A$ of $\mathcal{A}(G)$, let $G_A = \{g \in G | \alpha(g) = g, \alpha \in A\}$. It is easy to see that $G_A$ is a closed subgroup of $G$.

A locally compact space $X$ is called a homogeneous $G$-space if $G$ acts on $X$ transitively. Thus when $H$ is a closed subgroup of $G$, $G/H$ is a homogeneous $G$-space with the action of $G$ on $G/H$ by left translation. A regular Borel measure $m$ on $X$ is $G$-invariant if $m(gE) = m(E)$ for all $g \in G$ and all Borel subsets $E$ of $X$.
Lemma 1. Let $G$ and $G'$ be two locally compact groups and $X$ (resp. $X'$) be a homogeneous $G$-space (resp. $G'$-space). Let $\pi: G \to G'$ be an open and continuous epimorphism and $\eta: X \to X'$ be a continuous surjection such that $\eta(gx) = \pi(g)\eta(x)$ ($g \in G, x \in X$).

We have the following:

(i) $\eta$ is an open map.
(ii) If $X$ admits a finite $G$-invariant measure, then $X'$ admits a finite $G'$-invariant measure.
(iii) If $\eta$ is bijective (i.e. if the actions of $G$ on $X$ and of $G'$ on $X'$ are equivalent), then

(a) the converse of (ii) holds, and
(b) $X$ is compact if and only if $X'$ is compact.

Proof. (i) Let $U$ be a neighborhood of some $x$ in $X$; we show that $\eta(U)$ contains an open neighborhood of $\eta(x)$. Since $G$ and $G'$ are $\sigma$-compact, it follows that the mappings

$$
G \to X, \quad G' \to X',
$$

$$
g \mapsto gx, \quad g' \mapsto g'\eta(x),
$$

are both open and continuous (see e.g. [3, p. 7]). Let $V$ be an open neighborhood of 1 in $G$ such that $Vx \subseteq U$. Hence $\pi(V)\eta(x) = \eta(Vx)$ is an open neighborhood of $\eta(x)$ contained in $\eta(U)$.

(ii) Let $m$ be a finite $G$-invariant measure on $X$. Define $m'$ on $X'$: $m'(E) = m(\eta^{-1}(E))$ for every Borel set $E$ of $X'$. Then $m'$ is a finite regular Borel measure on $X'$. Now for any $g' \in G'$, there exists a $g \in G$ such that $\pi(g) = g'$ and $\eta^{-1}(g'E) = g\eta^{-1}(E)$. Therefore

$$
m'(g'E) = m(\eta^{-1}(g'E)) = m(g\eta^{-1}(E)) = m'(E).
$$

(iii) (b) is obvious since $\eta$ is now a homeomorphism. For (a): Let $m'$ be a finite $G'$-invariant measure on $X'$ and define $m$ on $X$: $m(E) = m'(\eta(E))$ for any Borel subset $E$ of $X$. It is easy to see that $m$ is a finite $G$-invariant regular Borel measure.

Lemma 2 [6, Lemma 2.5]. Let $H \subseteq F$ be closed subgroups of a locally compact group $G$ such that $G/H$ admits a finite invariant measure $m$. Then $G/F$ and $F/H$ admit finite invariant measures of which $m$ is a product.

Lemma 3 [2, Theorem 2]. If $G$ is a Lie group and $m(G/G_A)$ is finite, then $G/G_A$ is compact.

Lemma 4 [5, Theorem 2.3]. If $\pi: P \to P'$ is a continuous epimorphism of connected compact groups, then $\pi$ maps the center of $P$ onto the center of $P'$.

2. Proof of the theorem. First we show that it suffices to consider the case when $G$ is connected. $G_0$ is open so $G_0G_A$ is a closed subgroup of $G$. Hence, by Lemma 2, both $G/G_0G_A$ and $G_0G_A/G_A$ admit invariant measures. Since $G/G_0G_A$ is discrete, $G/G_0G_A$ is finite. On the other hand, the $G_0G_A$-space $G_0G_A/G_A$ is equivalent to the $G_0$-space $G_0/G_0 \cap G_A$ and it follows from Lemma 1 that $G_0/G_0 \cap G_A$ admits a finite invariant measure. But $G_0 \cap G_A$
\( (G_0)_A \) where \( A' = \{ \alpha_{G_0} | \alpha \in A \} \) is a subset of \( \mathfrak{A}(G_0) \). And so by assumption \( G_0 / G_0 \cap G_A \) is compact and, by Lemma 1, \( G_0 G_A / G_A \) is compact. Thus \( G / G_A \) compact follows. This completes the proof of the reduction to the case when \( G \) is connected.

From now on \( G \) is connected and we proceed to prove the theorem in four cases.

Case (i). \( K(G) = \{ 1 \} \).

Let \( P \) be a compact normal subgroup of \( G \) such that \( G / P \) is a Lie group. Since \( PK(G) \) is a compact normal subgroup containing \( K(G) \) and \( K(G) \) is maximal, it follows that \( P \subset PK(G) = K(G) = \{ 1 \} \). Therefore \( G \) is a Lie group. Thus by Lemma 3, \( G / G_A \) is compact.

Case (ii). \( K(G)_0 = \{ 1 \} \).

Let \( G' = G / K(G) \) and \( \pi: G \to G' \) be the projection. For any \( \alpha \in \mathfrak{A}(G) \), \( \alpha(K(G)) \) is again a compact normal subgroup of \( G \) and so as in Case (i), \( \alpha(K(G)) \subset K(G) \) (i.e. \( K(G) \) is characteristic in \( G \)). Hence \( \alpha \) induces an automorphism \( \alpha' \) of \( G' \) such that for any \( g \in G \), \( \alpha'(\pi(g)) = \pi(\alpha(g)) \). Let \( A' = \{ \alpha' | \alpha \in A \} \) and \( G' \) is characteristic in \( G \). Hence it follows from Case (i) that \( G' / G'_{A'} \) is compact.

Let \( H_A = \{ g \in G | \alpha(g)g_1^{-1} \in K(G), \alpha \in A \} \). It is obvious that \( H_A = \pi^{-1}(G'_{A'}) \) is a closed subgroup of \( G \). Define a mapping

\[ \eta: G / G_A \to G' / G'_{A'}, \quad \eta(g G_A) = \pi(g) G_{A'}. \]

It is easy to see that \( \eta \) is a continuous surjection and so, by Lemma 1, \( G' / G'_{A'} \) has a finite \( G' \)-invariant measure. Since the pull back of any compact normal subgroup of \( G' \) by \( \pi \) is a compact normal subgroup of \( G \) contained in \( K(G) \), it follows that \( K(G') = \{ 1 \} \). Hence it follows from Case (i) that \( G' / G'_{A'} \) is compact.

Then it is easy to see that \( \psi \) is a continuous bijection and so, by Lemma 1, \( G / H_A \) is compact. Hence for \( G / G_A \) to be compact, it remains to show that \( H_A / G_A \) is compact.

Since \( G_A \subset H_A \) and \( G / G_A \) has a finite \( G \)-invariant measure, it follows from Lemma 2 that \( H_A / G_A \) has a finite \( H_A \)-invariant measure. As \( G \) is connected, it is obvious that \( \mathfrak{A}(G)_{K(G)} \subset \mathfrak{A}(K(G))_0 \). Let \( \mathfrak{A}(K(G)_0) \) denote the subgroup of inner automorphisms of \( K(G) \) induced by elements of \( K(G)_0 \); then \( \mathfrak{A}(K(G)_0) = \mathfrak{A}(K(G))_0 \). Since \( K(G)_0 = \{ 1 \} \), \( [G, K(G)] = \{ 1 \} \). So for any \( g_1, g_2 \in H_A \), we have

\[ \alpha(g_1 g_2) (g_1^{-1} g_2^{-1}) = \alpha(g_1) (g_2^{-1} g_1^{-1} \alpha(g_2) g_1^{-1}). \]

Hence the mapping \( f: H_A \to K(G) \), \( f(g) = \alpha(g) g_1^{-1} \) is a homomorphism with \( G_A \) as its kernel. Hence \( G_A \) is normal in \( H_A \) and \( H_A / G_A \) is compact. This completes the proof of Case (ii).

Case (iii). The center \( Z \) of \( K(G)_0 \) is trivial.

Since \( K(G)_0 \) is characteristic in \( K(G) \) and \( K(G) \) is characteristic in \( G \), it follows that \( K(G)_0 \) is characteristic in \( G \). Let \( G' = G / K(G)_0 \) and \( \pi: G \to G' \) be the projection. Then as in Case (ii), \( A' \) induces a family \( A' \) of automorphisms...
of \( G' \) such that \( G'/G'A' \) admits a finite \( G' \)-invariant measure. Since it is easy to see that \( (K(G'))_0 = \{1\} \), it follows from Case (ii) that \( G'/G'A' \) is compact.

Let \( C = \{ g \in G \mid gkg^{-1} = k, k \in K(G)_0 \} \); then it follows from [4] that \( G = K(G)_0 C \). Here \( K(G)_0 \cap C = Z = \{1\} \) and \( [K(G), C] = \{1\} \). And so \( G = K(G)_0 \times C \) is a direct product and \( \pi_C : C \to G' \) is an isomorphism. Now let \( C_A = \{ c \in C \mid \alpha(c) = c, \alpha \in A \} \). We shall show that \( \pi(C_A) = G'A' \). It is obvious that \( \pi(C_A) \subset G'A' \). To see the converse inclusion, let \( g' \in G'A' \); then there is a unique \( c \in C \) such that \( \pi(c) = g' \). So for any \( \alpha \in A, \pi(c) = \alpha'(\pi(c)) = \pi(\alpha(c)) \) or \( \alpha(c)c^{-1} \in K(G)_0 \). But \( C \) is characteristic in \( G \), hence \( \alpha(c)c^{-1} \subset C \cap K(G)_0 \). Thus \( \alpha(c) = c \) or \( g' \in \pi(C_A) \). Therefore \( \pi_C \) induces a homeomorphism \( \eta : C/C_A \to G'/G'A' \), \( \eta(cC_A) = \pi(c)G'A' \). Hence \( C/C_A \) is compact.

Let \( K_A = \{ k \in K(G)_0 \mid \alpha(k) = k, \alpha \in A \} \) and define

\[
\psi : (K(G)_0/K_A) \times (C/C_A) \to G/G_A, \psi(kK_A, cC_A) = kcG_A.
\]

Here \( \psi \) is well defined, for if \( kk_k^{-1} \in K_A, cc^{-1} \in C_A \), then \( (kc)(k_1c_1)^{-1} = (kk_k^{-1})(cc^{-1}) \in K_A C_A \subset G_A \).

Let \( \psi_1 : G \to G/G_A \) be the continuous projection and \( \psi_2 : K(G)_0 \times C \to (K(G)_0/K_A) \times (C/C_A) \) be the open projection. Then \( \psi_1 = \psi \circ \psi_2 \) since \( G = K(G)_0 \times C \). Hence \( \psi \) is continuous and \( G/G_A \) is compact.

Case (iv). \( Z \neq \{1\} \).

Since \( K(G)_0 \) is characteristic in \( G \), so is \( Z \). Let \( G' = G/Z \) and \( \pi : G \to G' \) be the projection. Then as in Case (ii) \( A \) induces a family \( A' \) of automorphisms of \( G' \) such that \( G'/G'A' \) admits a finite \( G' \)-invariant measure. As \( Z \) is compact, therefore \( \pi(K(G)_0) = (K(G'))_0 \). Hence by Lemma 4, we have center of \( (K(G'))_0 = \pi(Z) \). But \( \pi(Z) = \{1\} \), therefore it follows from Case (iii) that \( G'/G'A' \) is compact.

Let \( H_A = \{ g \in G \mid \alpha(g)g^{-1} \in Z, \alpha \in A \} \). Then \( H_A = \pi^{-1}(G'A') \) is a closed subgroup of \( G \) containing \( G_A \). As in Case (ii), we have \( G/H_A \) compact. So for \( G/G_A \) to be compact, it remains to prove that \( H_A/G_A \) is compact. Now

\[
\overline{\mathcal{Z}}(G)_Z \subset \mathbb{V}(Z)_0 = \overline{\mathcal{Z}}(Z_0) = \{1\}
\]

where \( \overline{\mathcal{Z}}(Z_0) \) denotes the subgroup of inner automorphisms of \( Z \) induced by elements of \( Z_0 \). Therefore \( [Z, G] = \{1\} \) and analogous arguments as those in Case (ii) show that \( H_A/G_A \) is compact. This completes the proof of the theorem.

REFERENCES


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