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BALANCED VALUATIONS AND FLOWS IN MULTIGRAPHS

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ABSTRACT. A balanced valuation of a multigraph H is a mapping w of its vertex-set $V(H)$ into R such that $\forall S \subseteq V(H)$ the number of edges of H with exactly one vertex in S is greater than or equal to $|\Sigma_{v \in S} w(v)|$; we apply the theory of flows in networks to obtain known and new results on balanced valuations such as:

A cubic multigraph has chromatic index 3 if and only if it has a balanced valuation with values in $\{-2, +2\}$ (Bondy [5]).

Every planar cubic 2-edge connected multigraph has a balanced valuation with values in $\{-5/3, +5/3\}$.

Every planar 5-regular 4-edge connected multigraph has a balanced valuation with values in $\{-3, +3\}$.

I. Introduction.

1. *Basic definitions and notations.* A *multigraph* H (respectively: *directed multigraph* G) is a finite nonempty set $V(H)$ (respectively $V(G)$) of *vertices* together with a finite family $E(H)$ (respectively $A(G)$) of 2-element subsets of $V(H)$ called *edges* (respectively: of ordered pairs of elements of $V(G)$ called *arcs*).

When we replace each edge $e = \{v,v'\}$ $(v,v' \in V(H))$ of a multigraph H by an arc (v,v') or (v',v) we obtain a directed multigraph G (with $V(G) = V(H)$) which will be called *an orientation of H*.

For a directed multigraph G, for any $S \subseteq V(G)$, let
$\omega_G^+(S) = \{a \in A(G) | a = (v,v'), v \in S, v' \not\in S\}$.
$\omega_G^-(S) = \omega_G^+(V(G) - S)$.
$\omega_G(S) = \omega_G^+(S) \cup \omega_G^-(S)$.
For any vertex v of G:
The *outdegree* of v in G is $d_G^+(v) = |\omega_G^+(\{v\})|$.
The *indegree* of v in G is $d_G^-(v) = |\omega_G^-(\{v\})|$.
The *degree* of v in G is $d_G(v) = |\omega_G(\{v\})| = d_G^+(v) + d_G^-(v)$.
For a multigraph H, for any $S \subseteq V(H)$, let
$\omega_H(S) = \{e \in E(H) | |e \cap S| = 1\}$.
For any vertex v of H, the *degree* of v in H is $d_H(v) = |\omega_H(\{v\})|$. Other definitions not given here will be found in [1] and [2].

2. *Flows in directed multigraphs.* Let G be a directed multigraph with $|A(G)| = m \geqslant 1$ and $A(G) = \{a_1, \ldots, a_m\}$. A *flow in G with values* ϕ_i $(i = 1, \ldots, m)$ is a vector $\phi = (\phi_1, \ldots, \phi_m) \in Z^m$ such that

$$\forall S \subseteq V(G) \qquad \sum_{a_i \in \omega_G^+(S)} \phi_i - \sum_{a_i \in \omega_G^-(S)} \phi_i = 0.$$

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The flows in G form a submodule of Z^m. For every p and q in Z let $[p,q] = \{z \in Z | p \leqslant z \leqslant q\}$. Let $b_1, \ldots, b_m, c_1, \ldots, c_m$ be elements of Z with $b_i \leqslant c_i$ $\forall i \in [1,m]$. Then we have [1]

FEASIBLE FLOW THEOREM. *There is a flow* $\phi = (\phi_1, \ldots, \phi_m)$ *in* G *with* $\phi_i \in [b_i,c_i]$ $\forall i \in [1,m]$ *if and only if*

$$\forall S \subseteq V(G) \qquad \sum_{a_i \in \omega_G^+(S)} c_i - \sum_{a_i \in \omega_G^-(S)} b_i \geqslant 0.$$

3. *Flows and face colorings for planar multigraphs.*

PROPOSITION 1. *A planar directed multigraph G is face colorable with k colors ($k \geqslant 2$) if and only if there is a flow in G with values in*

$$[-k+1,-1] \cup [1,k-1].$$

REMARK. The two preceding equivalent properties imply that G has no acyclic arc (bridge).

SKETCH OF A PROOF. Through duality for planar multigraphs, flows become tensions or potential differences. Proposition 1 is a consequence of the following result: a multigraph G is k-colorable ($k \geqslant 2$) if and only if there is a potential difference in G with values in $[-k+1,-1] \cup [1,k-1]$, this result is essentially Minty's Theorem on colorings of multigraphs [3] interpreted with the help of the "Feasible Potential Difference Theorem".

An easy and well-known consequence of Proposition 1 is that a planar multigraph is face colorable with 2 colors if and only if it is eulerian.

We are then led to the following

DEFINITION. Let H be a multigraph. A $[p,q]$-*orientation of* H is an orientation G of H such that there is a flow in G with values in $[p,q]$; if such an orientation G exists, H will be said to be $[p,q]$-*orientable.*

Proposition 1 now becomes

PROPOSITION 2. *A planar multigraph H is face colorable with k colors ($k \geqslant 2$) if and only if it is $[1,k-1]$-orientable.*

This is our motivation for the study of the $[p,q]$-orientability property for general multigraphs; as we shall see in the next section, this property is of interest even in the nonplanar case.

II. [1,3]-orientability of cubic multigraphs and their chromatic index.

THEOREM 1. *Let H be a cubic multigraph. H is $[1,3]$-orientable if and only if it has chromatic index* 3.

PROOF. Suppose first that the edges of H are colored with 3 colors α, β and γ. Let G be any orientation of H, with every arc colored as the corresponding edge of H. It is easy to construct two flows ϕ' and ϕ'' in G such that

arcs colored with α or β have flow values $+1$ or -1 in ϕ', arcs colored γ have flow values 0 in ϕ'.

arcs colored with α or γ have flow values $+1$ or -1 in ϕ'', arcs colored β have flow values 0 in ϕ''.

Then $\phi' + 2\phi''$ is a flow in G with values in $[-3,-1] \cup [1,3]$; hence H is $[1,3]$-orientable.

Conversely let G be an [1,3]-orientation of H and ϕ a flow in G with values in [1,3]. The value of an edge of H will be defined as the value of ϕ in the corresponding arc of G. It will be easily seen that, due to the conservation of flow at each vertex, the edges of H with value 2 form a perfect matching of H.

Hence the edges of H with values 1 or 3 form a 2-factor F of H; the orientation G of H defines an orientation K of F. It will be easily checked that if we reverse each arc of K corresponding to an edge of value 3 we obtain an orientation K' of F such that:

$$\forall v \in V(H) = V(K') \qquad |d_{K'}^{+}(v) - d_{K'}^{-}(v)| = 2.$$

Hence F is bipartite and H has chromatic index 3. This completes the proof of Theorem 1.

REMARK. According to Proposition 2, Theorem 1 is a generalization of the following result of Heawood [4]: A cubic planar 2-edge connected multigraph is face colorable with four colors if and only if it has chromatic index 3.

III. **Balanced valuations of a multigraph.**

1. *Definitions.* Let H be a multigraph. A *balanced valuation* of H is a mapping w of $V(H)$ into R such that $\forall S \subseteq V(H)$ $|\Sigma_{v \in S} w(v)| \leqslant |\omega_H(S)|$. Bondy [5] has shown that a cubic graph H has chromatic index 3 if and only if there is a balanced valuation of H with values in $\{-2, +2\}$. We shall see that this result is a consequence of Theorem 1.

2. *Realization of a sequence of outdegrees by an orientation of a multigraph.*

PROPOSITION 3. *Let H be a multigraph with $V(H) = \{v_1, \ldots, v_n\}$. Let $k_1, \ldots, k_n$ be nonnegative integers. There is an orientation G of H such that $d_G^+(v_i) = k_i$ $\forall i \in [1,n]$ if and only if the valuation w of H defined by*

$$\forall i \in [1,n] \quad w(v_i) = 2k_i - d_H(v_i)$$

is balanced.

PROOF. w is balanced if and only if

$$\forall S \subseteq V(H) \quad -|\omega_H(S)| \leqslant 2 \sum_{v_i \in S} k_i - \sum_{v_i \in S} d_H(v_i) \leqslant |\omega_H(S)|.$$

For every $S \subseteq V(H)$ let H_S be the submultigraph of H induced by S; then

$$\sum_{v_i \in S} d_H(v_i) = 2|E(H_S)| + |\omega_H(S)|.$$

Hence w is balanced if and only if

$$\forall S \subseteq V(H) \quad |E(H_S)| \leqslant \sum_{v_i \in S} k_i \leqslant |E(H_S)| + |\omega_H(S)|.$$

It will be easily checked that this condition is equivalent to the following

$$\begin{cases} \text{(i)} \ \forall S \subseteq V(H) \ \Sigma_{v_i \in S} k_i \geqslant |E(H_S)|, \\ \text{(ii)} \ \Sigma_{i \in [1,n]} k_i = |E(H)|. \end{cases}$$

Hakimi [6] has shown that conditions (i) and (ii) are necessary and sufficient for H to have an orientation G with $d_G^+(v_i) = k_i$ $\forall i \in [1,n]$. This completes the proof.

3. *Nonnull flows and balanced valuations of a multigraph.*

THEOREM 2. *A multigraph H is $[p,q]$-orientable* $(0 < p < q)$ *if and only if there is a balanced valuation w of H such that*

$$\forall v \in V(H) \quad \exists k_v \in Z, \quad k_v \equiv d_H(v) \pmod 2\colon w(v) = (q+p)k_v/(q-p).$$

PROOF. Let G be an orientation of H. According to the Feasible Flow Theorem G is a $[p,q]$-orientation of H if and only if

$$\forall S \subseteq V(G) \begin{cases} q|\omega_G^+(S)| \geqslant p|\omega_G^-(S)|, \\ q|\omega_G^-(S)| \geqslant p|\omega_G^+(S)|. \end{cases}$$

(The second condition is the first condition where S is replaced by $V(G) - S$.) This is equivalent to

$$\forall S \subseteq V(G) \begin{cases} (q+p)(|\omega_G^-(S)| - |\omega_G^+(S)|) \leqslant (q-p)(|\omega_G^-(S)| + |\omega_G^+(S)|), \\ (q+p)(|\omega_G^+(S)| - |\omega_G^-(S)|) \leqslant (q-p)(|\omega_G^+(S)| + |\omega_G^-(S)|), \end{cases}$$

or

$$\frac{q+p}{q-p} \left| \, |\omega_G^+(S)| - |\omega_G^-(S)| \, \right| \leqslant |\omega_G(S)| = |\omega_H(S)| \quad \forall S \subseteq V(G).$$

But

$$\forall S \subseteq V(G) \quad |\omega_G^+(S)| - |\omega_G^-(S)| = \sum_{v \in S} d_G^+(v) - \sum_{v \in S} d_G^-(v)$$
$$= \sum_{v \in S} (2d_G^+(v) - d_H(v)).$$

Hence G is a $[p,q]$-orientation of H if and only if the valuation w of H defined by

$$\forall v \in V(H) \quad w(v) = \frac{q+p}{q-p}(2d_G^+(v) - d_H(v))$$

is balanced. The result now follows from Proposition 3 and from the fact that $(q+p)/(q-p) > 1$.

REMARK. The values of any balanced valuation w of a multigraph H are restricted by the condition: $\forall v \in V(H)$ $|w(v)| \leqslant d_H(v)$.

IV. **Applications.**

1. *Cubic multigraphs.*

PROPOSITION 4. *A cubic multigraph H is $[p,q]$-orientable* $(0 < p < q)$ *if and only if there is a balanced valuation of H with values in*

$$\{-(q+p)/(q-p), (q+p)/(q-p)\}.$$

PROOF. If $w(v) = (q+p)k_v/(q-p)$ with $|w(v)| \leqslant 3$ and $k_v \equiv 3 \pmod 2$, then $|k_v| = 1$; the result now follows from Theorem 2.

PROPOSITION 5 (BONDY [5]). *A cubic multigraph has chromatic index 3 if and only if it has a balanced valuation with values in* $\{-2, +2\}$.

This is a consequence of Theorem 1 and Proposition 4.

PROPOSITION 6. *Let* w *be a balanced valuation of a cubic multigraph* H *with values in* $\{-(q+p)/(q-p), (q+p)/(q-p)\}$; *let*

$$W^+ = \{v \in V(H) | w(v) > 0\}$$

and $W^- = V(H) - W^+$. *Then* $|W^+| = |W^-|$; *moreover* H_{W^+} *and* H_{W^-} *are forests, each component of which having at most* $[q/p] - 1$ *vertices* ($[q/p]$ *is the integer part of* q/p).

We omit the easy proof.

PROPOSITION 7. *A cubic multigraph is* [1,2]-*orientable if and only if it is bipartite.*

PROOF. According to Proposition 6, for $p = 1$ and $q = 2$, (W^+, W^-) form a coloring of H. Conversely, every bicoloring is of this type.

REMARK. Proposition 7 is a generalization of the well-known result [2] that a planar cubic multigraph is face colorable with 3 colors if and only if it is bipartite.

PROPOSITION 8. *Every planar cubic* 2-*edge connected multigraph has a balanced valuation with values in* $\{-5/3, 5/3\}$.

This is exactly the Five Color Theorem for planar multigraphs, formulated with the help of Propositions 2 and 4.

PROPOSITION 9. *The vertices of every planar cubic* 2-*edge connected multigraph can be colored with* 2 *colors in such a way that*

The two colors form two sets of vertices of the same cardinality.

There is no monochromatic elementary cycle and no monochromatic tree on 4 *vertices.*

This is a direct consequence of Propositions 8 and 6.

2. *Other applications.* Any result or problem about colorings of planar graphs could be stated in a "balanced valuation" formulation. For instance:

FOUR COLOR PROBLEM. Does every 2-edge connected planar multigraph H have a balanced valuation w with $w(v) \equiv 2d_H(v) \pmod 4$ $\forall v \in V(H)$?

REMARK. The Four Color Problem can be restricted to cubic multigraphs, or to 5-regular multigraphs; in both cases the corresponding balanced valuations have values in $\{-2, +2\}$.

PROPOSITION 10. *A planar* 2-*edge connected multigraph* H *is face colorable with* 3 *colors if and only if it has a balanced valuation* w *with* $w(v) \equiv 3d_H(v)$ $\pmod 6$ $\forall v \in V(H)$.

REMARKS. From a theorem of Grötzsch [7] this is true for 4-edge connected planar multigraphs.

For 5-regular or 7-regular multigraphs the corresponding balanced valuations have values in $\{-3, +3\}$.

Propositions 10 and 3 allow us to write:

PROPOSITION 11. *A planar* 2-*edge connected multigraph* H *is face colorable with* 3 *colors if and only if there is an orientation* G *of* H *such that*:

$$\forall v \in V(G) \quad d_G^+(v) - d_G^-(v) \equiv 0 \pmod 3.$$

It can be seen that Heawood's result [4] relating vertex assignments modulo 3 and edge colorations with 3 colors in a planar cubic multigraph H is a consequence of Proposition 11 applied to the "radial graph" of H.

3. *Further problems.* So far we have been concerned only with $[p,q]$-orientability for $p = 1$. An infinite class of problems arises when we consider other values of p. For instance:

CONJECTURE. Every planar 2-edge connected multigraph is [2,7]-orientable. This would be better than the Five Color Theorem but weaker than the Four Color Theorem.

NOTES ADDED IN PROOF (SEPTEMBER 1975). (1) It can be proved, using a result of Tutte (Theorem 5.44 in [8]), that a multigraph H is $[1, k-1]$-orientable ($k \geqslant 2$) if and only if every orientation G of H has a colour-cycle (nowhere-zero flow) over Z_k (the cyclic group of order k); see [9] and [10].

Hence $[1, k-1]$-orientability can be studied using Tutte's theory of dichromatic polynomials [9].

(2) Theorem 1 has been proved by Minty in [11]; an elementary proof appears in [10].

(3) Tutte has conjectured that:

every 2-edge-connected multigraph is [1, 4]-orientable (see [9]);

every 4-edge-connected multigraph is [1, 2]-orientable.

It is proved in [10] that:

every 2-edge-connected multigraph is [1, 7]-orientable;

every 4-edge-connected multigraph is [1, 3]-orientable.

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