

p TH POWERS OF DISTINGUISHED SUBFIELDS

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ABSTRACT. Let $k \supset s \supset k_0$ be fields of characteristic $p \neq 0$, k/k_0 finitely generated and s a distinguished subfield. The field $k_0(k^{(n)}) = \{x \in k \mid x^{p^t} \in k_0(k^{p^{n+t}})\}$ for some $t \geq 0$ has $k_0(s^{p^n})$ as a distinguished subfield and is maximal in k with respect to this property. Let \bar{k}_0^s and \bar{k}_0 be, respectively, the separably algebraic closure and the algebraic closure of k_0 in k . Then $\bar{k}_0 = \bigcap_k k_0(k^{(n)})$. Also $\bar{k}_0^s = \bar{k}_0$ if and only if $k_0(k^{p^n}) \supset k_0(k^{(n)})$ for some n . For n large $k_0(k^{(n)}) = \bar{k}_0(k^{p^n})$. The sequence $\{[k_0(k^{(n)}): k_0]_i\}_n$ is decreasing, descending from $[k: k_0]_i$ to $[\bar{k}: k_0]_i$ in a finite number of steps. Examples are given which show: (1) that $k_0(k^{(n)})$ may have distinguished subfields not of the form $k_0(s^{p^n})$; and, (2) how to construct k/k_0 so that the sequence $\{[k_0(k^{(n)}): k_0]_i\}$ has preassigned values.

1. Introduction. Let k be a field, a finitely generated extension of a subfield k_0 having characteristic $p \neq 0$. In 1947 Dieudonné introduced the concept distinguished subfield [1]. An intermediate field s is a distinguished subfield of k/k_0 if (1) s is separable over k_0 and (2) $k \subset k_0^*(s)$ where k_0^* is the perfect closure of k_0 . By (2) s is a maximal separable intermediate field. Dieudonné showed that if s is a distinguished subfield then $[k: s] = [k: k_0]_i$ the order of inseparability of k over k_0 as defined by A. Weil [5, p. 22]. He also demonstrated that if s is a maximal separable intermediate field it need not be true that $[k: s] = [k: k_0]_i$. Kraft has shown [4] that an intermediate field s separable over k_0 is distinguished if and only if $[k: s] = [k: k_0]_i$.

In this paper we investigate p th powers of distinguished subfields beginning with the observation (Theorem 2) that for each $n \geq 0$ there is a unique intermediate field $k_0(k^{(n)})$ having the property that if s is a distinguished subfield of k then $k_0(s^{p^n})$ is a distinguished subfield of $k_0(k^{(n)})$ and $k_0(k^{(n)})$ is maximal with respect to this property. Also, as the notation suggests, $k_0(k^{(n)})$ is independent of the choice of s . Theorem 5 asserts that $\bigcap_n k_0(k^{(n)})$ is the algebraic closure \bar{k}_0 of k_0 in k as $\bigcap_n k_0(k^{p^n})$ is the separable algebraic closure, \bar{k}_0^s , of k_0 in k [2, Theorem 7.2, p. 273]. Theorem 9 states that for n large $k_0(k^{(n)}) = \bar{k}_0(k^{p^n})$. Corollaries 6 and 8 give necessary and sufficient conditions in terms of $k_0(k^{(n)})$ that $k_0 = \bar{k}_0^s$.

The latter part of the paper is primarily concerned with the connection between $[\bar{k}_0: k_0]_i$, $[k: k_0]_i$ and the sequence $\{[k_0(k^{(n)}): k_0]_i\}_{n \geq 0}$. Corollary 10 states that, for large n , $[k_0(k^{(n)}): k_0]_i = [\bar{k}_0: k_0]_i$ and Theorem 11 states that $[k_0(k^{(n)}): k_0]_i$ is a decreasing function of n . The final result, Theorem 12, asserts that k/\bar{k}_0 is separable if and only if $[k_0(k^{(n)}): k_0]_i = [\bar{k}_0: k_0]_i$ for all $n \geq 0$.

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Following Theorem 12 two examples are provided. The first demonstrates that in general $k_0(k^{(n)})$ has distinguished subfields not of the form $k_0(s^{p^n})$. The second illustrates how, given a descending sequence $n_0 \geq n_1 \geq n_2 \geq \dots \geq n_t = n_{t+1} = \dots$ of nonnegative integers, one can construct a finitely generated extension k/k_0 such that $[k_0(k^{(j)}): k_0]_i = p^{n_j}$ for all $j \geq 0$.

II. p th powers of distinguished subfields. Throughout this paper k will be a field, a finitely generated extension of a field k_0 of characteristic $p \neq 0$ and s will denote a distinguished intermediate field. Thus, by definition, s/k_0 is separable and $k \subset k_0^*(s)$ where k_0^* is the perfect closure of k_0 [1, p. 13]. We denote $\{x \in k \mid \text{for some } t \geq 0, x^{p^t} \in k_0(k^{p^{t+n}})\}$ by $k_0(k^{(n)})$ and observe that $k_0(k^{(n)})$ is a subfield containing $k_0(k^{p^n})$.

We begin with a result first observed by Kraft [4, Theorem 3, p. 113], a result which is in fact an immediate consequence of the definition of distinguished subfield.

1. PROPOSITION. *An intermediate field s separable over k_0 is a distinguished subfield of k over k_0 if and only if for some positive integer n , $k_0(s^{p^n}) = k_0(k^{p^n})$.*

PROOF. If $k_0(s^{p^n}) = k_0(k^{p^n})$ then $k \subset k_0^{p^{-n}}(s)$ and s is distinguished. Conversely, if s is distinguished then $k \subset k_0^{p^{-n}}(s)$ for some n . Hence $k^{p^n} \subset k_0(s^{p^n})$ which implies $k_0(k^{p^n}) = k_0(s^{p^n})$.

2. THEOREM. *Let s be a distinguished subfield of k/k_0 . For each positive integer n the following is true.*

- (1) $k_0(s^{p^n})$ is a distinguished subfield of $k_0(k^{(n)})/k_0$.
- (2) If $k_0(s^{p^n})$ is a distinguished subfield of l over k_0 with $l \subset k$ then $l \subset k_0(k^{(n)})$.

PROOF. By definition, $k_0(s^{p^n})$ is a distinguished subfield of $l \subset k$ if and only if $l \subset e = k_0^*(s^{p^n}) \cap k$. We need only show that $k_0^*(s^{p^n}) \cap k = k_0(k^{(n)})$. If $x \in e$ then $x^{p^t} \in k_0(s^{p^{n+t}})$ for some t and hence $x \in k_0(k^{(n)})$. Conversely, if x is in $k_0(k^{(n)})$ then, by definition, $x^{p^t} \in k_0(k^{p^{t+n}})$ for some t . Hence, for $m > t$, $x^{p^m} \in k_0(k^{p^{m+n}})$. For m sufficiently large $k_0(k^{p^{m+n}}) = k_0(s^{p^{m+n}})$ and we conclude that $x \in k_0^*(s^{p^n}) \cap k$.

3. COROLLARY. *If k is separable over k_0 then $k_0(k^{(n)}) = k_0(k^{p^n})$ for all $n \geq 0$.*

PROOF. In this case $k = s$ and $k_0(k^{p^n})$ is a maximal separable subfield of $k_0(k^{(n)})$ over k_0 . But $k_0(k^{(n)})$ is separable over k_0 and so $k_0(k^{(n)}) = k_0(k^{p^n})$.

The converse of Corollary 3 is not true as the following example proves. Let P be a perfect field of characteristic $p \neq 0$, and let x, y , and z be indeterminates. We define $k_0 = P(x, y)$, $s = k_0(z)$ and $k = s(u)$ where $u^p = x + yz^p$. Clearly, s is a distinguished subfield.

CONTENTION. $k_0(k^{(1)}) = k_0(k^p) = k_0(s^p) = k_0(z^p)$.

PROOF. The first is the only equality in the above chain which is not immediate. $k_0(k^{(1)}) = \{z \in k \mid z^{p^i} \in k_0(k^{p^{i+1}}) \text{ for some } i\} = \{z \in k \mid z^{p^2} \in k_0(k^{p^2})\}$ by Corollary 3 since $k_0(k^p)/k_0$ is separable. Hence

$$k_0(k^{(1)}) = k_0^{p^{-1}}(k^p) \cap k = P(x^{p^{-1}}, y^{p^{-1}}, z^p) \cap P(x, y, z, x^{p^{-1}} + y^{p^{-1}}z)$$

$$\supset P(x, y, z^p) = k_0(z^p).$$

We replace x and y by x^p and y^p in the above and we have either $k_0(k^{(1)}) = k_0(z^p)$ or there are polynomials $f(X, Y)$ and

$$g(X, Y) \in P(x^p, y^p, z^p)[X, Y]$$

such that $f \notin P(x^p, y^p, z^p)[X^p, Y^p]$ but $f(x, y) = g(z, x + yz)$. Write

$$g(z) = g_i(z)(x + yz)^i + g_{i-1}(z)(x + yz)^{i-1} + \dots + g_0(z).$$

The coefficient of $y^i z^i$ in the above is $g_i(z)$. Hence $g_i(z)y^i z^i \in P(x, y, z^p)$ or $g_i(z) = g'_i z^{p-i}$ with $g'_i \in P(x^p, y^p, z^p)$. But the term

$$ig'_i xy^{i-1} z^{i-1} \notin P(x, y, z^p)$$

and $g(z)$ has no other term in xy^{i-1} with coefficients in $P(x^p, y^p, z)$. It follows that $g(z, x + yz)$ does not have the form $f(x, y)$. This proves the above contention. Using Lemma 4 below and induction on n we have $k_0(k^{(n)}) = k_0(k^{(n-1)})^{(1)} = k_0(k^{p^{n-1}})^{(1)} = k_0(k^{p^n})$; the last equality being given by Lemma 3 since $k_0(k^p)/k_0$ is separable.

For simplicity's sake we accept the abuse of notation in the next result.

4. LEMMA. $k_0(k^{(n+1)}) = k_0(k^{(n)})^{(1)}$ for $n \geq 0$.

PROOF. If x is in $k_0(k^{(n)})^{(1)}$ then for some $t \geq 0$, $x^{p^t} = \sum a_i b_i^{p^{t+1}}$ with $a_i \in k_0$, $b_i \in k_0(k^{(n)})$. For some r , and each i , $b_i^{p^r} \in k_0(k^{p^{n+r}})$. Thus $x^{p^{t+r}} = \sum a_i^{p^r} (b_i^{p^r})^{p^{t+1}} \in k_0(k^{p^{n+r+t+1}})$ or $x \in k_0(k^{(n+1)})$. Conversely, if x is in $k_0(k^{(n+1)})$ then for t large $x^{p^t} \in k_0(k^{p^{n+t+1}}) \subset k_0(k^{(n)})^{p^{t+1}}$ since $k_0(k^{p^n}) \subset k_0(k^{(n)})$. So x is in $k_0(k^{(n)})$ and $x^{p^t} \in k_0(k_0(k^n))^{p^{t+1}}$, that is, $x \in k_0(k^{(n)})^{(1)}$.

If x is in \bar{k}_0 , the algebraic closure of k_0 in k , then for some $t \geq 0$, $x^{p^t} \in \bar{k}_0^s$, the separably algebraic closure of k_0 in k . Hence x^{p^t} is in $k_0(k^{p^{n+t}})$ for all n since $k_0((\bar{k}_0^s)^{p^{n+t}}) = \bar{k}_0^s$. Thus $x \in \bigcap_n k_0(k^{(n)})$. Conversely, if $x \in \bigcap_n k_0(k^{(n)})$ then for any $n > 0$ there is an integer t such that $x^{p^t} \in k_0(k^{p^{t+n}})$ or $x \in k_0^*(k^{p^n})$. Thus $x \in \bigcap_n k_0^*(k^{p^n})$. Applying the fact that $\bigcap_n k_0(k^{p^n})$ is algebraic over k_0 [2, Theorem 7.2, p. 273] to $k_0^*(k)$ we conclude that x is algebraic over k_0^* and hence over k_0 . We have proved the following.

5. THEOREM. The algebraic closure of k_0 in k is $\bigcap_n k_0(k^{(n)})$.

Compare Theorem 5 with the theorem of Heerema and Deveney referred to above that $\bigcap_n k_0(k^{p^n})$ is the separably algebraic closure of k_0 in k . Let r be the inseparability exponent for k over k_0 , that is r is the least positive integer such that $k_0(k^{p^r})$ is separable over k_0 .

6. COROLLARY. The algebraic closure \bar{k}_0 of k_0 in k is separable over k_0 if and only if for some n

$$(7) \quad k_0(k^{(n)}) = k_0(k^{p^n}) = k_0(s^{p^n}).$$

If (7) holds for n then $n \geq r$ and (7) holds for $t > n$.

PROOF. If (7) holds for n then clearly $n \geq r$ and $k_0(k^{(n)})$ is separable over k_0 . Hence, for $j \geq 0$, $k_0(k^{(n+j)}) = k_0(k^{(n)})^{(j)} = k_0(k_0(k^{p^n}))^{p^j}$, by Corollary 3. It follows that (7) holds for all n . Also $k_0(k^{(n)}) = \bigcap_n k_0(k^{p^n})$. By

Theorem 5 and the result cited below Theorem 5 we have $\bar{k}_0^s = \bigcap_n k_0(k^{(n)}) = \bigcap_n k_0(k^{p^n}) = \bar{k}_0$.

If $\bar{k}_0^s \neq \bar{k}$ then clearly $k_0(k^{(n)}) \neq k_0(k^{p^n})$ for $n \geq r$ since $k_0(k^{(n)})$ is not separable over k_0 for any n .

The following provides a weaker condition equivalent to \bar{k}_0/k_0 separable.

8. COROLLARY. *The algebraic closure \bar{k}_0 of k_0 in k is separable over k_0 if and only if, for some $n > 0$, $k_0(k^p) \supset k_0(k^{(n)})$.*

PROOF. If $k_0(k^p) \supset k_0(k^{(n)})$, then, since $\bar{k}_0 \cap k_0(k^p) = k_0(\bar{k}_0^p)$ and $k_0(k^{(n)}) \supset \bar{k}_0$ we have $k_0(\bar{k}_0^p) = \bar{k}_0$ which is equivalent to \bar{k}_0/k_0 separable. The converse follows from Corollary 6. Theorem 5 suggests the following result.

9. THEOREM. *For n sufficiently large $k_0(k^{(n)}) = \bar{k}_0(k^{p^n})$.*

PROOF. By Theorem 5 $\bar{k}_0(k^{p^n}) \subset k_0(k^{(n)})$ for all $n \geq 0$ and by Corollary 6 $\bar{k}_0(k^{(n)}) = \bar{k}_0(k^{p^n})$ for large n . Since $k_0(k^{(n)}) \subset \bar{k}_0(k^{(n)})$ we have $\bar{k}_0(k^{p^n}) \subset k_0(k^{(n)}) \subset \bar{k}_0(k^{p^n})$ for n large which yields the theorem.

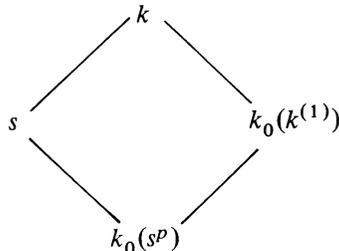
10. COROLLARY. *For large n , $[k_0(k^{(n)}): k_0]_i = [\bar{k}_0: k_0]_i$.*

PROOF. For $n \geq r$, $\bar{k}_0(k^{p^n}) = \bar{k}_0 \otimes_{\bar{k}_0^s} k_0(k^{p^n})$ since \bar{k}_0/\bar{k}_0^s is purely inseparable and $k_0(k^{p^n})/k_0$ is separable [3, Theorem 21, Part (1), p. 197]. Thus $[\bar{k}_0(k^{p^n}): k_0(k^{p^n})] = [\bar{k}_0: k_0]_i$. But, by Theorem 9, $k_0(k^{(n)}) = \bar{k}_0(k^{p^n})$ for large n . The result now follows with the observations that $k_0(k^{p^n}) = k_0(s^{p^n})$ for $n \geq r$ and $[k_0(k^{(n)}): k_0]_i = [k_0(k^{(n)}): k_0(s^{p^n})]$.

Dieudonné showed that $[k: s] = [k: k_0]_i$ as defined by Weil. The next result asserts that $[k_0(k^{(n)}): k_0]_i$ decreases monotonely with n from $[k: k_0]_i$ for $n = 0$ to $[\bar{k}_0: k_0]_i$ for large n .

11. THEOREM. $[k_0(k^{(n+1)}): k_0(s^{p^{n+1}})] \leq [k_0(k^{(n)}): k_0(s^{p^n})]$.

PROOF. By Lemma 4 it is only necessary to prove the result for $n = 0$. Consider the following diagram



Let $X = \{x_1, \dots, x_r\}$ be a separating transcendence basis for s over k_0 and assume X p -dependent over $k_0(k^{(1)})$. Let $f(x_1, \dots, x_r) = 0$ be a polynomial over $k_0(k^{(1)})$ of degree $< p$ in each x_i . Choose t so that the p^t th power of each coefficient of f is in $k_0(k^{p^{t+1}})$. If f is nontrivial $f^{p^t}(x_1^{p^t}, \dots, x_r^{p^t}) = 0$ exhibits a p -dependence among $x_1^{p^t}, \dots, x_r^{p^t}$ in $k_0(k^{p^t})$ over k_0 . However, a p -basis X for s over k_0 has the property that X^{p^n} is p -independent in $k_0(s^{p^n})$ over k_0 for all n since s/k_0 is separable. Since $k_0(s^{p^n}) = k_0(k^{p^n})$ for n large we have X^{p^n} p -independent in $k_0(k^{p^n})/k_0$ for large n and thus for all $n \geq 0$, a fact

Dieudonné used to construct distinguished subfields [1]. Thus f is trivial and X is p -independent over $k_0(k^{(1)})$, $[k: k_0(k^{(1)})] \geq [s: k_0(s^p)]$ and, by the above diagram $[k: s] \geq [k_0(k^{(1)}): k_0(s^p)]$.

12. THEOREM. *The following are equivalent.*

- (i) k/\bar{k}_0 is separable.
- (ii) $[k: k_0]_i = [\bar{k}_0: k_0]_i$.
- (iii) $[k_0(k^{(n)}): k_0]_i = [\bar{k}_0: k_0]_i$, for all $n \geq 0$.
- (iv) $[k_0(k^{(n)}): k_0]_i = [k_0(k^{(m)}): k_0]_i$ for all $n, m \geq 0$.

PROOF. To show (i) \Leftrightarrow (ii) we observe that $\bar{k}_0(s) = \bar{k}_0 \otimes_{\bar{k}_0} s$ and hence $\bar{k}_0(s) = k$ if and only if $[k: s] = [\bar{k}_0: \bar{k}_0^s]$. By Theorem 11, (ii) implies (iii) and (ii) is given by (iii) for $n = 0$. By Theorems 11 and 9 (iv) implies (iii). The remaining implication is immediate.

III. Two examples. The first example illustrates the fact that not all distinguished subfields of $k_0(k^{(1)})$ are of the form $k_0(s^p)$. It is derived from an example due to Dieudonné [1, p. 13].

EXAMPLE 1. Given a perfect field P of characteristic $p \neq 2$ and indeterminates $x, y,$ and z , we let $k_0 = P(x, y), s = k_0(z)$ and $k = s(v)$ where $v^p = x + yz^{p^2}$. Then s is a distinguished subfield of $k/k_0, k_0(s^p) = k_0(z^p), k_0(k^p) = k_0(z^p, v^p)$ and $k_0(k^{(1)}) \supset k_0(z^p, v)$ since $k_0(k^{(1)}) \supset k_0(k^p)$ and $v^p \in k_0(k^{p^2})$. Since $k_0(k^{(1)}) \neq k$, i.e., $k_0(s^p)$ is not a distinguished subfield of k , we have $k_0(k^{(1)}) = k_0(z^p, v)$. Now, $h = k_0(vz^p, z^{p^2})$ is separable over k_0 since z^{p^2} satisfies the equation $yX^2 + xX - (vz^p)^p = 0$. This and the fact that $k_0(k^{(1)})$ is purely inseparable of degree p over $k_0(vz^p, z^{p^2})$ implies that $k_0(vz^p, z^{p^2})$ is a distinguished subfield of $k_0(k^{(1)})/k_0$. Since $v \notin k_0(k^p)$ we conclude that $k_0(vz^p, z^{p^2})$ is not of the form $k_0(s^p)$.

EXAMPLE 2. Let t be a positive integer and let $n_0 \geq n_1 \geq \dots \geq n_t = n_{t+1} = \dots = n$ be a sequence of nonnegative integers. The following is an example of an extension k/k_0 such that $[k_0(k^{(j)}): k_0]_i = p^{n_j}$ for $j \geq 0$. It is related to the example following Corollary 3 and Example 1. We use a perfect field P and sets of indeterminates $\{w\}$ and $\{x_{i,j}, y_{i,j}, z_{i,j}\}_{i=0, j=1}^{t-1, q_i}$ where $q_i = n_i - n_{i+1}$ for $0 \leq i < t$ and there are no $x_{i,j}$ for i such that $q_i = 0$. Let $k_0 = P(\{x_{i,j}, y_{i,j}\}_{i,j}, w), s = k_0(\{z_{i,j}\}_{i,j})$ and $k = s(\{v_{i,j}\}_{i,j}, w^{p^{-n}})$ where $v_{i,j}^p = x_{i,j} + y_{i,j}z_{i,j}^{p^{q_i}}$. We note that s is a distinguished subfield of k/k_0 and $[k: s] = p^{n_0}$. It is shown below that $k_0(k^{(1)}) = k_0(\{z_{i,j}, v_{i+1,j}\}_{i,j}, w^{p^{-n}})$. Since $k_0(s^p) = k_0(\{z_{i,j}\})$ it follows that

$$[k_0(k^{(1)}): k_0(s^p)] = p^{\sum\{q_i | j > 0\} + n} = p^{n_1}.$$

Since $k_0(k^{(1)})/k_0$ has the same form as k/k_0 we use the fact that $k_0(k^{(j+1)}) = k_0(k^{(j)})^{(1)}$ to conclude that $[k_0(k^{(j)}): k_0]_i = p^{n_j}$ for $j \geq 0$.

We observe first that $k_0(k^{p^n}) = k_0(s^{p^n})$ and thus, by Corollary 3,

$$k_0(k^{(1)}) = \{u \in k | u^{p^n} \in k_0(k^{1+p^n})\} = k_0^{p^{-n}}(k^p) \cap k.$$

Let $V_m = \{v_{i,j}\}_{i=m, j=1}^{t-1, q_i}$ and define X_m, Y_m, Z_m similarly.

CONTENTION. $k_0^{p^{-n}}(k^p) \cap k = k_0(s^p)(V_1, w^{p^{-n}})$.

PROOF. The claim is that

$$(13) \quad \begin{aligned} P(X_0^{p^{-n}}, Y_0^{p^{-n}}, Z_0^p, w^{p^{-n}}) \cap P(X_0, Y_0, Z_0, V_0, w^{p^{-n}}) \\ = P(X_0, Y_0, Z_0^p, V_1, w^{p^{-n}}). \end{aligned}$$

Let $h = P(X_0, Y_0, Z_0^p, V_1, w^{p^{-n}})$ and assume u to be in the left side of (13). Then u is a polynomial $f(X_0^{p^{-n}}, Y_0^{p^{-n}})$ with coefficients in h and is a polynomial $g(Z_0, V_0)$ with coefficients in h . Let $i = 0$ and fix j arbitrarily. Regard f as a polynomial in $x_{0,j}^{p^{-n}}$ and $y_{0,j}^{p^{-n}}$ with coefficients in $h_{0,j}$ which is h extended by the remaining $x_{i,j}^{p^{-n}}, y_{i,j}^{p^{-n}}, z_{i,j}$. Similarly we view g as a polynomial in $z_{0,j}$ and

$$v_{0,j} = x_{0,j}^{p^{-1}} + y_{0,j}^{p^{-1}} z_{0,j}$$

over $h_{0,j}$. Simplifying the notation in the obvious way we have $f(x^{p^{-n}}, y^{p^{-n}}) = g_i(z)(x^{p^{-1}} + y^{p^{-1}}z)^i + \cdots + g_0(z)$. The quantity $y^{ip^{-1}}z^i$ occurs only in the i th term of g as a polynomial in $x^{p^{-1}} + y^{p^{-1}}z$. Hence $g_i(z) = z^{p^{-i}}g^*$ where $g^* \in h_{0,j}$. The quantity $x^{p^{-1}}y^{(i-1)p^{-1}}z^{i-1}$ also occurs only in the i th term and therefore has coefficient $ig_i^* z^{p^{-i}}$. It follows that $i = 0$ in view of the equality $f(x^{p^{-n}}, y^{p^{-n}}) = g(z, x^{p^{-1}} + y^{p^{-1}}z)$. Hence $u \in h_{0,j}$ for all j . In like manner one establishes the corresponding result for each i, j pair from which fact (13) follows.

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