AN ANALOGUE OF SOME INEQUALITIES OF P. TURAN
CONCERNING ALGEBRAIC POLYNOMIALS
HAVING ALL ZEROS INSIDE [−1, +1]

A. K. VARMA

Abstract. Let \( P_n(x) \) be an algebraic polynomial of degree \( < n \) having all its zeros inside \([-1, +1]\); then we have
\[
\int_{-1}^{1} P_n^2(x) \, dx > \frac{n}{2} \int_{-1}^{1} P_n^2(x) \, dx.
\]
The result is essentially best possible. Other related results are also proved.

Let \( H_n \) be the set of all polynomials whose degree does not exceed \( n \), i.e., polynomials of the form
\[ P(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n. \]
Here the coefficients \( c_0, c_1, \ldots, c_n \) are arbitrary real numbers. The following inequalities on algebraic polynomials are well known.

**Theorem A.** Let \( P_n(x) \in H_n \); then we have
\[
\max_{-1 < x < +1} (1 - x^2) P_n^2(x) \leq n^2 \max_{-1 < x < +1} P_n^2(x),
\]
and
\[
\max_{-1 < x < +1} |P_n'(x)| \leq n^2 \max_{-1 < x < +1} |P_n(x)|.
\]

(\( P_n'(x) \) stands for the derivative of \( P_n(x) \).)
(1.1) is due to S. N. Bernstein [1] and (1.2) to A. A. Markov [2]. In this work we are concerned with the following beautiful theorem of P. Turán [4].

**Theorem B.** Let \( P_n(x) \) be an algebraic polynomial of degree \( < n \) having all its zeros inside \([-1, +1]\); then we have
\[
\max_{-1 < x < +1} |P_n'(x)| > \frac{n^{1/2}}{6} \max_{-1 < x < +1} |P_n(x)|.
\]
(1.3) was later sharpened by Janos Eröd [2], who proved

**Theorem C.** Under the assumptions of Theorem B we have for \( P_n(x) \in H_n \)
\[
\max_{-1 < x < +1} \left| \frac{P_n'(x)}{P_n(x)} \right| > \frac{n}{2} \quad \text{if } n = 2, 3,
\]
\[
> \frac{n}{\sqrt{n - 1}} \left( 1 - \frac{1}{n - 1} \right)^{(n-2)/2}, \quad n \text{ even, } n \geq 4,
\]
\[
> \frac{n^2}{(n - 1) \sqrt{n + 1}} \left( 1 - \frac{\sqrt{n + 1}}{n - 1} \right)^{(n-3)/2} \left( 1 + \frac{1}{\sqrt{n + 1}} \right)^{(n-1)/2},
\]
\text{if } n > 5 \text{ and odd.}
\]

Further, this result is best possible.

Inequalities on polynomials analogous to (1.2) in the norm
\[
\|f\|_{L^1[-1, +1]}^2 = \int_{-1}^{1} f^2(x) \, dx
\]
were proved by E. Schmidt [5]. See also the contributions of Einar Hille, G. Szegö and J. D. Tamarkin [4].

**Theorem D** [E. Schmidt]. Let us denote
\[
M_n^2 = \max_{f \in H_n} \left[ \int_{-1}^{1} f^2(x) \, dx / \int_{-1}^{+1} f^2(x) \, dx \right];
\]
then for \( n \geq 5 \) \((-6 < R < 13)\)
\[
(1.5) \quad M_n = \left( \frac{n + \frac{1}{2}}{\pi} \right)^2 \left( 1 - \frac{\pi^2 - 3}{12(n + \frac{1}{2})^2} + \frac{R}{(n + \frac{1}{2})^4} \right)^{-1}.
\]

In view of Theorems B and D it is natural to ask: If \( P_n \in H_n \) and all zeros of \( P_n(x) \) are inside \([-1, +1]\), then how small can the expression \( \int_{-1}^{+1} P_n^2(x) \, dx / \int_{-1}^{+1} P_n^2(x) \, dx \) be? The following theorem concerns the above question.

**Theorem 1.** Let \( P_n(x) \in H_n \) and assume that all its zeros are inside \([-1, +1]\); then we have
\[
(1.7) \quad \int_{-1}^{+1} P_n^2(x) \, dx > \frac{n}{2} \int_{-1}^{+1} P_n^2(x) \, dx.
\]

This result is best possible in the sense that there exists a polynomial \( p_0(x) \) of degree \( n \) having all zeros inside \([-1, +1]\) and for which
\[
(1.8) \quad \int_{-1}^{+1} P_0^2(x) \, dx / \int_{-1}^{+1} P_0^2(x) \, dx = \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n - 1)}, \quad n > 1.
\]

The proof of Theorem 1 is based on

**Theorem 2A.** Let \( f_n(x) \) be an algebraic polynomial of degree \( \leq n \) having all zeros inside \([-1, +1]\); then we have
\[
(1.9) \quad \frac{n}{2} \leq \int_{-1}^{+1} f_n^2(x)(1 - x^2) \, dx / \int_{-1}^{+1} f_n^2(x) \, dx,
\]
with equality only for \( f_n(x) = (1 + x)^p(1 - x)^q \), \( p + q = n \).
Theorem 2B. Let \( f_n(x) \) be an algebraic polynomial of degree \( \leq n \); then we have
\[
\int_{-1}^{+1} f_n^2(x)(1 - x^2) \, dx / \int_{-1}^{+1} f_n^2(x) \, dx \leq n(n + 1),
\]
with equality only for \( f_n(x) = cP_n(x) \) (\( P_n(x) \) being the Legendre polynomial of degree \( n \)).

(1.10) may be regarded as analogous to (1.1) in the \( L_2 \) norm.

2. Since \( f_n(x) \) has all zeros inside \([-1, +1]\) we may write
\[
(2.1) \quad f_n(x) = \prod_{k=1}^{n} (x - x_k)
\]
where \(-1 \leq x_k \leq +1, k = 1, 2, \ldots, n.

Professor P. Turan [6] observed that
\[
(2.2) \quad f_n'(x) = f_n(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)}
\]
and
\[
(2.3) \quad f_n^2(x) - f_n(x)f_n''(x) = f_n^2(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)^2}.
\]

On multiplying (2.3) by \((1 - x^2)\) and using (2.2) we obtain
\[
2(1 - x^2)f_n^2(x) - \frac{d}{dx} \left( (1 - x^2)f_n(x)f_n'(x) \right) = (1 - x^2)f_n^2(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)^2} + 2xf_n(x)f_n'(x)
\]
\[
= f_n^2(x) \sum_{k=1}^{n} \frac{(1 - x^2) + 2x(x - x_k)}{(x - x_k)^2}.
\]
Therefore
\[
2(1 - x^2)f_n^2(x) - nf_n^2(x) - \frac{d}{dx} \left\{ (1 - x^2)f_n(x)f_n'(x) \right\}
\]
\[
= f_n^2(x) \sum_{k=1}^{n} \frac{1 - x^2 + 2x(x - x_k) - (x - x_k)^2}{(x - x_k)^2}
\]
\[
= f_n^2(x) \sum_{k=1}^{n} \frac{(1 - x_k^2)}{(x - x_k)^2} \geq 0.
\]

On integrating both sides from \(-1\) to \(1\) we obtain
\[
(2.4) \quad 2 \int_{-1}^{+1} (1 - x^2)f_n^2(x) \, dx - nf \int_{-1}^{+1} f_n^2(x) \, dx > 0,
\]
with equality only for \( f_n(x) = (1 + x)^p(1 - x)^q \), \( p + q = n \). But this in turn implies (1.9). This proves Theorem 2A.

The proof of Theorem 2B depends on...
(2.6) \[ \int_{-1}^{1} p_i(x) p_j(x) \, dx = 0, \quad i \neq j, \]
\[ = \frac{2}{(2i + 1)}, \quad i = j, \]
and
(2.7) \[ \int_{-1}^{1} (1-x^2)p_i'(x)p_j'(x) \, dx = 0, \quad i \neq j, \]
\[ = 2i(i+1)/(2i+1), \quad i = j, \]
where \( p_j(x) \) denotes the Legendre polynomial of degree \( j \) in \( x \). Writing
\[ f_n(x) = \sum_{i=0}^{n} \lambda_i p_i(x), \quad f_n'(x) = \sum_{i=0}^{n} \lambda_i p_i'(x) \]
and using (2.6) and (2.7) we obtain
\[ \int_{-1}^{1} f_n^2(x)(1-x^2) \, dx = \sum_{i=1}^{n} \frac{\lambda_i^2(i+1)}{(2i+1)} \]
\[ \leq n(n+1). \]
Obviously equality occurs only if \( f_n(x) = cP_n(x) \) (\( P_n(x) \) being Legendre polynomial of degree \( n \)). This proves Theorem 2B.

**Proof of Theorem 1.** Since \( 1-x^2 \leq 1, -1 < x < +1 \), we have
\[ (1-x^2)p_n^2(x) < p_n^2(x) \quad \text{for} -1 < x < +1. \]
Therefore
(2.8) \[ \int_{-1}^{1} (1-x^2)p_n^2(x) \, dx < \int_{-1}^{1} p_n^2(x) \, dx. \]
But from Theorem 2A we have
(2.9) \[ \int_{-1}^{1} (1-x^2)p_n^2(x) \, dx \geq \frac{n}{2} \int_{-1}^{1} p_n^2(x) \, dx. \]
From (2.8) and (2.9) it follows that
\[ \int_{-1}^{1} p_n^2(x) \, dx \geq \frac{n}{2} \int_{-1}^{1} p_n^2(x) \, dx. \]
This proves (1.7). It remains to prove (1.8). Let \( f_n(x) = p_0(x) = (1-x^2)^m, \)
\( 2m = n; \) then
\[ \int_{-1}^{1} p_0^2(x) \, dx = \int_{-1}^{1} (1-x^2)^m \, dx = \frac{2\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \]
and
\[ \int_{-1}^{1} p_0^2(x) \, dx = \frac{2\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \]
\[ = 4n^2\int_{0}^{\pi/2} \sin^{2n-3}\theta \cos^2\theta \, d\theta \]
\[ = n^2\frac{\Gamma(n-1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}. \]
Therefore
\[
\frac{\int_{-1}^{1} P_0^2(x) \, dx}{\int_{-1}^{1} P_2(x) \, dx} = \frac{n(n + \frac{1}{2})}{2(n - 1)} = \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n - 1)}, \quad n > 1.
\]
This proves Theorem 1 as well.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601