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A COMPARISON THEOREM[1]

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ABSTRACT. A general comparison theorem is extracted from the proof of a recent comparison theorem of Leighton and Oo Kian Ke. The result is applied to a comparison theorem for focal points.

The results of this note are inspired by a recent theorem of Leighton and Oo Kian Ke [2]. The method of these authors can be used to prove a comparison theorem for general homogeneous boundary conditions.

THEOREM. *Let*

$$u'' + p_1(t)u = 0, \qquad y'' + p_2(t)y = 0,$$

where the $p_i(t)$ are positive and continuous. If

(1) $u(t) \geqslant 0, y(t) \geqslant 0$ *in* $\tau_0 \leqslant t \leqslant \tau_1$,

(2) $u(\tau_0) = y(\tau_0)$, $u'(\tau_0)=y'(\tau_0)$,

(3) $\cos \beta u(\tau_1) - \sin \beta u'(\tau_1) = \cos \beta y(\tau_1) - \sin \beta y'(\tau_1) = 0$,

(4) *there exists* $\sigma \in (\tau_0, \tau_1)$ *so that* $p_1(t) > p_2(t)$ *for* $t < \sigma$, $p_1(t) < p_2(t)$ *for* $t > \sigma$, *then* $u(\tau_1) \leqslant y(\tau_1)$ *and* $|u'(\tau_1)| \leqslant |y'(\tau_1)|$. *The inequalities are strict unless* $\cos \beta \sin \beta = 0$ *in which cases the nontrivial inequalities are still strict.*

The proof follows [2] closely. Since

$$\lim_{t \downarrow \tau_0} \frac{u''(t)}{y''(t)} = \frac{p_1(\tau_0)}{p_2(\tau_0)} > 1$$

and $u''(t) \leqslant 0$ for $t \in [\tau_0, \tau_1]$ it follows that $u'(t) < y'(t)$ and $u(t) < y(t)$ in some interval $\tau_0 < t < \tau_0 + \varepsilon$. We prove that $u(t) < y(t)$ in (τ_0, τ_1). For

$$W(t) = u(t)y'(t) - u'(t)y(t)$$

we have $W(\tau_0) = W(\tau_1) = 0$. Since

$$W'(t) = [p_1(t) - p_2(t)]u(t)y(t)$$

changes sign only once in $[\tau_0, \tau_1]$, by Rolle's theorem $W(t) > 0$ in (τ_0, τ_1). If $u(t) \geqslant y(t)$ somewhere in (τ_0, τ_1), by continuity there exists a first value t_0 where $u(t_0) = y(t_0)$ and $t_0 > \tau_0$. At t_0,

$$W(t_0) = [y'(t_0) - u'(t_0)]u(t_0) > 0 \quad \text{and} \quad y'(t_0) > u'(t_0).$$

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But this is impossible since the graph of y is above the graph of u in $(\tau_0, \tau_0 + \varepsilon)$ and at the first intersection the graph of y cannot have greater slope than that of u. Our assertion is proved and it follows that $u(\tau_1) \leqslant y(\tau_1)$.

For $\sigma < t < \tau_1$ we have

$$u''(t)/y''(t) = p_1(t)u(t)/p_2(t)y(t) < 1$$

and $y''(t) < u''(t)$. By integration,

$$u'(\tau_1) - y'(\tau_1) > u'(t) - y'(t) \quad \text{for} \quad t \in (\sigma, \tau_1). \tag{1}$$

Case 1. If $\cos \beta = 0$, $u'(\tau_1) = y'(\tau_1) = 0$ and $u'(t) < y'(t)$ for $t \in (\sigma, \tau_1)$. From (1) we have

$$u(\tau_1) = u(t) + \int_t^{\tau_1} u'(s)\, ds < y(t) + \int_t^{\tau_1} y'(s)\, ds = y(\tau_1)$$

for t close to τ_1. This inequality is strict as asserted in the theorem.

Case 2. If $\sin \beta = 0$, $u(\tau_1) = y(\tau_1) = 0$ and, from (1),

$$u'(\tau_1) - y'(\tau_1) > (y(t) - u(t))/(\tau_1 - t) > 0.$$

Since the derivatives are negative, $|u'(\tau_1)| < |y'(\tau_1)|$. This is a result of [2].

Case 3. If $\cos \beta \sin \beta \neq 0$, assume that $u(\tau_1) = y(\tau_1)$. As in Case 2 it follows that $u'(\tau_1) > y'(\tau_1)$. Since, however, $u'(\tau_1) = u(\tau_1)\cot \beta$ and $y'(\tau_1) = y(\tau_1)\cot \beta$, we must have $u'(\tau_1) = y'(\tau_1)$. The contradiction shows that $u(\tau_1) \neq y(\tau_1)$ and $u'(\tau_1) \neq y'(\tau_1)$.

The first focal point $f(t_0)$ of an equation $x'' + p(t)x = 0$ is the first zero $> t_0$ of the derivative of a nontrivial integral of the equation which itself vanishes at t_0. Similarly, the second focal point $g(t_0)$ is defined as the first zero $> t_0$ of an integral of the equation whose derivative vanishes at t_0. The first conjugate point $c(t_0) = g[f(t_0)]$ is the first zero $> t_0$ of a nontrivial integral of the equation that vanishes at t_0.

PROPOSITION. *If the hypotheses of the theorem hold for* $u(\tau_0) = y(\tau_0) = 0$, $u'(\tau_1) = y'(\tau_1) = 0$, *i.e., for the first focal points* f_u *and* f_y *of the equations we have* $f_u(\tau_0) = f_y(\tau_0) = \tau_1$ *then*

$$f_u'(\tau_0) > f_y'(\tau_0)$$

or

$$f_u(\tau_0 - \varepsilon) < f_y(\tau_0 - \varepsilon), \qquad f_u(\tau_0 + \varepsilon) > f_y(\tau_0 + \varepsilon)$$

for sufficiently small ε.

PROOF. Since [1, p. 113, (7)],

$$f_u'(t_0) = \{p[f_u(t_0)]\}^{-1}\{u'(t_0)/u[f_u(t_0)]\}^2,$$

we have by the theorem

$$\frac{f_u'(\tau_0)}{f_y'(\tau_0)} = \frac{p_2(\tau_1)}{p_1(\tau_1)}\left[\frac{y(\tau_1)}{u(\tau_1)}\right]^2 > 1.$$

The theorem of Leighton and Oo Kian Ke [2] is the corresponding result for $c(t)$. No similar result holds for $g(t)$ as can be seen from [1, p. 113, (8)]. The derivation formulae have also been first obtained by Leighton.

References

1. O. Borůvka, *Lineare Differentialtransformationen* 2. *Ordnung*, Hochschulbücher für Mathematik, Band 67, VEB Deutscher Verlag der Wissenschaften, Berlin, 1967. MR**38** #4743.

2. W. Leighton and W. Oo Kian Ke, *A comparison theorem*, Proc. Amer. Math. Soc. **28**(1971), 185–188. MR**42** #8002.

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