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A PERTURBATION THEOREM FOR COMPLETE SETS OF COMPLEX EXPONENTIALS

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ABSTRACT. The purpose of this note is to show that the completeness of a set of complex exponentials $\{e^{i\lambda_n t}\}$ in $L^2(-\pi,\pi)$ is preserved whenever the λ_n are subjected to a suitable "lifting".

There is an extensive literature on the completeness of sets of complex exponentials $\{e^{i\lambda_n t}\}$ (see, for example, [1]–[8], and the references therein). In this note, we show that completeness is preserved in $L^2(-\pi,\pi)$ whenever the λ_n are subjected to a suitable "lifting".

THEOREM. *Let* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *be two sequences of points lying in a fixed horizontal strip and suppose that* $\operatorname{Re}\lambda_n = \operatorname{Re}\mu_n$. *If* $\{e^{i\lambda_n t}\}$ *is complete in* $L^2(-\pi,\pi)$, *then so too is* $\{e^{i\mu_n t}\}$.

PROOF. By making a suitable translation, we may assume that $\lambda_n\mu_n \neq 0$. Suppose that the set $\{e^{i\mu_n t}\}$ is not complete in $L^2(-\pi,\pi)$. Then there exists a function f_0 in $L^2(-\pi,\pi)$ not equivalent to zero such that

$$\int_{-\pi}^{\pi} f_0(t)e^{i\mu_n t}\,dt = 0 \qquad (n = 1,2,\ldots).$$

Let us denote by H the Paley-Wiener space of entire functions F of exponential type π for which

$$\|F\| = \left\{\int_{-\infty}^{\infty} |F(x)|^2\,dx\right\}^{1/2} < \infty.$$

If we set

$$F_0(z) = \int_{-\pi}^{\pi} f_0(t)e^{izt}\,dt,$$

then F_0 belongs to H, is not identically zero, and $F_0(\mu_n) = 0$ for each μ_n. We may suppose in addition that $F_0(0) = 1$. This is clear if $F_0(0) \neq 0$, while if F_0 has a zero of order m at the origin, then dividing F_0 by a suitable multiple of z^m produces the desired function.

Let

$$F_n(z) = F_0(z)\prod_{k=1}^{n}\frac{z-\lambda_k}{z-\mu_k}\frac{\mu_k}{\lambda_k} \qquad (n = 1,2,\ldots).$$

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Then $F_n \in H$, $F_n(0) = 1$, and $F_n(\lambda_k) = 0$ $(k = 1, 2, \ldots, n)$. We are going to show that the norms $\|F_n\|$ are uniformly bounded in n. By the Paley-Wiener representation for functions in H, we have

$$F_n(z) = \int_{-\pi}^{\pi} f_n(t)e^{izt}\,dt \qquad \text{with } f_n \text{ in } L^2(-\pi,\pi).$$

But then

$$\int_{-\pi}^{\pi} f_n(t)e^{izt}\,dt = \frac{z-\lambda_n}{z-\mu_n}\frac{\mu_n}{\lambda_n}\int_{-\pi}^{\pi} f_{n-1}(t)e^{izt}\,dt,$$

and it was shown by Levinson [2, p. 10] that

$$f_n(t) = \frac{\mu_n}{\lambda_n}\left[f_{n-1}(t) + i(\lambda_n - \mu_n)e^{-i\mu_n t}\int_{-\pi}^{\pi} f_{n-1}(x)e^{i\lambda_n x}\,dx\right].$$

Since $\sup|\lambda_n - \mu_n| < \infty$, obvious estimates yield

$$\|f_n\| \leqslant A|\mu_n/\lambda_n|\,\|f_{n-1}\|,$$

where A is independent of n. Therefore,

$$\|f_n\| \leqslant A\|f_0\| \prod_{k=1}^{n}\left|\frac{\mu_k}{\lambda_k}\right|,$$

and it remains only to estimate the products $\prod_{k=1}^{n}|\mu_k/\lambda_k|$. From the conditions on $\{\lambda_n\}$ and $\{\mu_n\}$ it follows that

$$\left|\frac{\mu_k}{\lambda_k}\right|^2 = 1 + \frac{(\mathrm{Im}\,\mu_k)^2 - (\mathrm{Im}\,\lambda_k)^2}{(\mathrm{Re}\,\lambda_k)^2 + (\mathrm{Im}\,\lambda_k)^2} \leqslant 1 + B/|\lambda_k|^2,$$

where B is independent of k. Therefore, for all n,

$$\prod_{k=1}^{n}\left|\frac{\mu_k}{\lambda_k}\right| \leqslant \prod_{k=1}^{n}\left(1+\frac{B}{|\lambda_k|^2}\right)^{1/2} \leqslant \prod_{k=1}^{\infty}\left(1+\frac{B}{|\lambda_k|^2}\right)^{1/2}$$
$$\leqslant \exp\left[\frac{B}{2}\sum_{k=1}^{\infty}\frac{1}{|\lambda_k|^2}\right].$$

Now, F_0 is entire of exponential type, and hence of order no larger than 1. Therefore, its exponent of convergence is also at most 1, and in particular, the series $\sum 1/|\mu_n|^2$ is convergent. It follows that the series $\sum 1/|\lambda_n|^2$ is also convergent, and we conclude that $\sup\|f_n\| < \infty$. Since the Fourier transform is an isometry, the norms $\|F_n\|$ are uniformly bounded. But H is a functional Hilbert space, and therefore a subsequence of $\{F_n\}$ will converge weakly to a function G in H for which $G(\lambda_n) = 0$ $(n = 1, 2, \ldots)$ and $G(0) = 1$. Writing

$$G(z) = \int_{-\pi}^{\pi} g(t)e^{izt}\,dt,$$

with g in $L^2(-\pi,\pi)$, we conclude that the sequence $\{e^{i\lambda_n t}\}$ is *not* complete in

$L^2(-\pi, \pi)$, contrary to assumption. The contradiction establishes the theorem.

ADDED IN PROOF. Ray Redheffer has informed me that this result, with a different proof, appeared in J. Elsner's doctoral dissertation (Georg-August Univ., Göttingen, 1969).

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