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THE EXISTENCE OF DUAL MODULES

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ABSTRACT. In this note we show that a Noetherian module has a dual module if and only if it satisfies $AB5^*$. A connection between completeness and $AB5^*$ is also established.

In this note we relate completeness, quasi-completeness, the $AB5^*$ condition, and duality. The main result is that a Noetherian R-module has a dual module if and only if it satisfies $AB5^*$.

Throughout this note R will denote a commutative ring with identity and all modules will be unitary. The terms local and semilocal will carry the Noetherian hypothesis. We use $L(A)$ or $L_R(A)$ to denote the lattice of R-submodules of A.

An R-module B is said to be a *dual* of an R-module A if there exists an order reversing lattice isomorphism θ: $L(A) \to L(B)$ satisfying $\theta(JN) = \theta(N)$: J for all R-submodules N of A and all ideals J of R.

Any R-module A satisfies the so-called $AB5$ condition: for any submodule B and any ascending chain $\{B_\alpha\}$ of submodules of A, $B \cap (\bigcup_\alpha B_\alpha) = \bigcup_\alpha (B \cap B_\alpha)$. A satisfies the dual condition $AB5^*$ if for any submodule B and any descending chain $\{B_\alpha\}$ of submodules, it follows that $B + (\bigcap_\alpha B_{\alpha-}) = \bigcap_\alpha (B + B_\alpha)$. Not every module satisfies $AB5^*$; for example, Z, the integers, does not. However, any module having a dual necessarily satisfies $AB5^*$. We show that for Noetherian modules, the converse is also true. We first show that the condition $AB5^*$ is closely related to completeness.

Let R be a semilocal ring with Jacobson radical J and let A be a finitely generated R-module. If A is complete in the J-adic topology, it is well known [6] that A satisfies the condition

(*) For any descending chain $\{B_n\}_{n=1}^{\infty}$ of submodules of A and any integer k, there exists an integer $n(k)$ such that $B_{n(k)} \subseteq (\bigcap_{n=1}^{\infty} B_n) + J^k A$.

A finitely generated module over a semilocal ring will be called quasi-complete if it satisfies (*).

The first theorem relates the concepts of quasi-completeness and $AB5^*$.

THEOREM 1. *Let R be a semilocal ring and A a finitely generated R-module. Then A satisfies $AB5^*$ if and only if it is quasi-complete.*

PROOF. Suppose A satisfies $AB5^*$. Let $\{B_n\}_{n=1}^{\infty}$ be a countable descending chain of submodules of A and let k be a fixed integer. Then

$$J^k A + \bigcap_{n=1}^{\infty} B_n = \bigcap_{n=1}^{\infty} (J^k A + B_n) = J^k A + B_{n(k)}$$

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for some integer $n(k)$ since A satisfies $AB5^*$ and A/J^kA is Artinian. Thus $B_{n(k)} \subseteq \bigcap_{n=1}^{\infty} B_n + J^kA$, so A is quasi-complete. Conversely, suppose that A is quasi-complete. Let $\{B_n\}_{n=1}^{\infty}$ be a countable descending chain in A, $B = \bigcap_{n=1}^{\infty} B_n$, and let C be a submodule of A. We show that $C + B = \bigcap_{n=1}^{\infty}(C + B_n)$. For fixed k, by quasi-completeness, there exists an integer $n(k)$ such that $B_{n(k)} \subseteq B + J^kA$, and hence $C + B_{n(k)} \subseteq C + B + J^kA$. We may assume $n(k) \to \infty$ as $k \to \infty$. Hence

$$\bigcap_{n=1}^{\infty}(C + B_n) = \bigcap_{n=1}^{\infty}(C + B_{n(k)}) \subseteq \bigcap_{k=1}^{\infty}(B + C + J^kA) = B + C$$

by the Krull Intersection Theorem. The reverse containment is always true. The result now follows since any chain in A is countable [2].

We next relate completeness and quasi-completeness. Let R be semilocal with Jacobson radical J and let A be a finitely generated R-module. $L(A)$, the lattice of submodules of A has a natural metric d defined on it by $d(C, D) = 2^{-n}$ if $C + J^nA = D + J^nA$ but $C + J^{n+1}A \neq D + J^{n+1}A$. The next theorem is due to E. W. Johnson [3].

THEOREM 2. *Let R be a semilocal ring and A a finitely generated R-module. Then the following are equivalent*:

(1) *the metric d on $L(A)$ is complete,*

(2) *A is quasi-complete,*

(3) *the map $L_R(A) \to L_{\hat{R}}(\hat{A})$ given by $N \to \hat{R}N$ (where $\hat{}$ denotes the J-adic completion) is surjective (and hence a lattice isomorphism).*

We remark that while any complete module is quasi-complete, a quasi-complete module need not be complete. For example, any $D \vee R$ is quasi-complete. More generally a one-dimensional local domain is quasi-complete if and only if it is analytically irreducible. The ring $k[X, Y]_{(X,Y)}$, k a field, is not quasi-complete.

The main theorem requires the following

LEMMA. *Let R be a Noetherian ring and A a finitely generated R-module satisfying $AB5^*$. Then* $\operatorname{Supp}(A)$ *contains only finitely many maximal elements; actually each $P \in$* $\operatorname{Ass}(A)$ *is contained in a unique maximal element of* $\operatorname{Supp}(A)$.

PROOF. Since $\operatorname{Ass}(A)$ is finite, the second statement implies the first. For $P \in \operatorname{Ass}(A)$, R/P is isomorphic to a submodule of A and hence satisfies $AB5^*$ as an R-module and hence as a ring. Thus we are reduced to showing that a Noetherian domain R satisfying $AB5^*$ must be local. Suppose not, say P and Q are distinct maximal ideals in R. Now $\{P^n\}_{n=1}^{\infty}$ is a descending chain of ideals in R and $\bigcap_{n=1}^{\infty} = 0$ by the Krull Intersection Theorem. Hence $Q + \bigcap_{n=1}^{\infty} P^n = Q$. However, for every n, $Q + P^n = R$, so $\bigcap_{n=1}^{\infty}(Q + P^n) = R$. Thus R must be local.

Finally, we require the theory of duality between Noetherian and Artinian modules over a complete local (or semilocal) ring given by Matlis [4] and [5]. (Also see [7] for an introduction into injective modules and duality.) Briefly, let R be a complete semilocal ring with Jacobson radical J. There is a perfect duality between Noetherian and Artinian R-modules given by the functor $\operatorname{Hom}_R(\_\,, E(R/J))$ where $E(R/J)$ is the injective envelope of R/J. Also for R

semilocal, but not necessarily complete, and for A an Artinian $\hat{R}$-module, the R-submodules and $\hat{R}$-submodules coincide and hence A is also Artinian as an R-module.

THEOREM 3. *For a finitely generated module A over a Noetherian ring R, the following are equivalent*:

(1) *A has a dual,*

(2) *A satisfies $AB5^*$,*

(3) *A is quasi-complete as an $\bar{R} = R/\operatorname{ann}(A)$ - module.*

PROOF. It is clear that (1) implies (2). Suppose A satisfies $AB5^*$. Then A satisfies $AB5^*$ as an $\bar{R}$ -module. By the previous lemma, $\bar{R}$ is semilocal. By Theorem 1, A is quasi-complete as an $\bar{R}$ -module. It remains to show that (3) implies (1). By change of rings, it suffices to show that A has an $\bar{R}$ -module dual. Thus we may replace R by $\bar{R}$ and assume that R is semilocal. By Theorem 2, the map $L_R(A) \to L_{\hat{R}}(\hat{A})$ given by $N \to \hat{R}N$ is a lattice isomorphism which preserves scalar product (i.e., $\hat{R}(JN) = J(\hat{R}N)$). Now as an $\hat{R}$-module, $\hat{A}$ has a dual, namely, $B = \operatorname{Hom}_{\hat{R}}(\hat{A}, E(\hat{R}/\hat{J}))$. Since B is Artinian as an $\hat{R}$-module, it follows that the R-submodules of B coincide with the $\hat{R}$-submodules of B. Hence B is actually an R-module dual of A.

We have shown that a Noetherian module has a dual if and only if it satisfies $AB5^*$. The hypothesis that the module be Noetherian cannot be deleted. Any Artinian module satisfies $AB5^*$; however, it is easily seen that the abelian group Z_{p^∞} (p a prime) does not have a Z-module dual. In fact, a result of Baer [1] states that an abelian group has a dual if and only if it is torsion and every primary component is finitely generated.

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