SHORTER NOTES

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A REMARK ON THE FIRST NEIGHBOURHOOD RING
OF A NOETHERIAN COHEN-MACAULAY LOCAL RING OF
DIMENSION ONE

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Abstract. There is an isomorphism between the first neighbourhood ring of a noetherian Cohen-Macaulay local ring $A$ of dimension one and the ring of endomorphisms of a large power of its maximal ideal.

Let $A$ be a noetherian Cohen-Macaulay local ring of dimension one and $m$ be its maximal ideal.

An element $a$ of $m'$ is superficial of degree $t$ if, for every large integer $n$, $m^n a = m^{n+t}$. The following results are well known: every superficial element is regular; for every large integer $t$, there exists a superficial element of degree $t$ [4]. The first neighbourhood ring $R$ of $A$ is the subring $\{(x/y) | x \in m', y \text{ superficial of degree } t\}$ of the total quotient ring $K$ of $A$. For every large integer $n$, the product $Rm^n = m^n$ [2, 12.1]. Let $v$ be the least such $n$.

Let $\text{End}_A(m^n)$ denote the algebra of $A$-endomorphisms of $m^n$. There is a sequence

$$A \subset \text{End}_A(m) \subset \cdots \subset \text{End}_A(m^n) \subset \cdots.$$ (1)

Theorem. 1. The integer $v$ is the least integer $n$ such that $xm^n = m^{n+t}$ for every superficial element $x$, where $t$ is the degree of $x$.

2. For every integer $n \geq v$, the ring $\text{End}_A(m^n) = \text{End}_A(m^n)$ and there exists an isomorphism $F$ of $A$-algebras of $\text{End}_A(m^n)$ onto $R$ such that $F(\text{Hom}_A(m^n,m^{n+1}))$ is the ideal $Rm$ of $R$.

Proof. 1. Let $x$ be a superficial element of degree $t$. Then $\text{length}(m^n/xm^n) = te$ where $e$ is the multiplicity of $A$ [2, 12.5]. If $xm^n = m^{n+t}$, then

$$\text{length}(m^n/xm^n) = \sum_{i=0}^{t-1} \text{length}(m^{n+i}/m^{n+i+1}) = te.$$
As \( \text{length}(m^n/m^{n+1}) \) is less than \( e \), we must have \( \text{length}(m^n/m^{n+1}) = e \). Then \( n \geq n \) \([2, 12.10]\).

On the other hand, \( x \) is superficial of degree \( t \) if and only if \( Rx = Rm^t \). As \( Rm^n = m^n \), \( Rxm^n = Rm^tm^n \) and so \( xm^n = m^{n+t} \).

2. Let \( t \) be an integer such that, for every integer \( s \geq t \), there exists in \( m^t \) a superficial element of degree \( s \). Let \( b \in m^t \) be a superficial element of degree \( t \). If \( k \) is a large integer, \( a = bk \) is superficial of degree \( s = kt \) and \( Ra = m^s \). Suppose \( n \geq n \). Then \( m^na = m^{n+s} \) by 1. If \( c \) is superficial of degree \( n + s \), then \( c = ad \) where \( d \in m^n \) is superficial of degree \( n \).

Define the homomorphism \( F: \text{End}_A(m^n) \to R \) by \( F(\phi) = \phi(d)/d \).

For every \( z \in m^n \) and \( \phi \in \text{End}_A(m^n) \), we have \( \phi(zd) = z\phi(d) = d\phi(z) \) and so \( \phi(z) = (\phi(d)/d)z \). So \( F \) is one to one. On the other hand since \( Ra = m^s \), every \( \lambda \in R \) is \( x/a \) where \( x \in m^s \). But \( am^n = m^{n+s} \); hence for every \( z \in m^n \), \( xz \) belongs to the ideal \( am^n \), so \( \lambda z \) belongs to \( m^n \). Define \( \phi \) by \( \phi(z) = \lambda z \). Then \( F(\phi) = \lambda \) and \( F \) is onto.

If \( \phi \in \text{Hom}_A(m^n, m^n+1) \), then \( \phi(d) \in m^{n+1} = Rm^{n+1} \). As \( d \) is superficial of degree \( n \), we have \( Rm^n = Rd \) and so \( \phi(d) \in Rd m \) and \( F(\phi) = \phi(d)/d \) belongs to \( Rm \).

Conversely, if \( \alpha \in Rm \), write \( \alpha = \sum \lambda_i e_i \) where \( \lambda_i \in R \) and \( e_i \in m \) to see that the element of \( \text{End}_A(m^n) \) defined by \( \phi(z) = \alpha z \) belongs to \( \text{Hom}_A(m^n, m^{n+1}) \).

References


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