PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

THE PREFRATTINI RESIDUAL

H. BECHTELL

ABSTRACT. The structure of a prefrattini subgroup in a finite solvable group is related to the structure of the prefrattini residual, that is, the residual for the formation of solvable nC-groups. In particular this structure can be identified with the system normalizers of a lower nilpotent series in the prefrattini residual. Moreover, the prefrattini subgroups form a complete conjugate class within this residual and also contain the system normalizers of the residual.

Notation is standard and may be found in [8] except for one deviation and that is $G = [A]B$ represents the group splitting over the normal subgroup A by the subgroup B in the group G. Only *finite solvable* groups are to be considered.

It will be assumed that the reader is familiar with the basic properties of Frattini subgroups (see [4] or [8]), the concept of the prefrattini subgroups introduced by W. Gaschütz [5], and the relationship between a prefrattini subgroup and a Sylow system given by T. Hawkes [7].

1. **The prefrattini residual.** A *solvable nC-group* is a solvable group that splits over each normal subgroup and the collection of all such groups is a formation $\mathfrak{F}$ (see [1]). That the $\mathfrak{F}$-residual $G_{\mathfrak{F}}$ can be more explicitly defined is due to a result of W. Gaschütz [5, Satz 6.6], namely, the prefrattini subgroup of a solvable group G is trivial if and only if G is an nC-group. Accordingly,

(1.1) $G_{\mathfrak{F}}$ is the normal closure of the prefrattini subgroups in a solvable group G.

For this reason, $G_{\mathfrak{F}}$ will be called the *prefrattini residual* of the group G.

From the definition of $\mathfrak{F}$, $\Phi(G) \subseteq G_{\mathfrak{F}}$ and $(G/\Phi(G))_{\mathfrak{F}} = G_{\mathfrak{F}}/\Phi(G)$ for which $\Phi(G)$ denotes the Frattini subgroup of the group G. If $\Phi(G) = 1$, then a prefrattini subgroup avoids the Fitting subgroup $F(G)$. A consequence is the following result:

(1.2) In a solvable group G, $G_{\mathfrak{F}}$ is nilpotent if and only if $G_{\mathfrak{F}} = \Phi(G)$.

Denote the residual for the formation of nilpotent groups in a group H by $K_\infty(H)$. The next conclusion follows from (1.2).

(1.3) If W is a prefrattini subgroup of a solvable group G, then $G_{\mathfrak{F}} = K_\infty(G_{\mathfrak{F}})W$.

$\mathfrak{F}$ admits another characterization. Denote the collection of solvable groups having the property that each homomorphic image has a trivial Frattini subgroup by $\mathfrak{P}$. With the aid of Satz 6.6 of [5] and (1.2), it can be established that $G \in \mathfrak{P}$ if and only if $G \in \mathfrak{F}$, that is, $\mathfrak{P} = \mathfrak{F}$.

If a solvable group $G \in \mathfrak{F}$, then the normal subgroups of G are in $\mathfrak{F}$ (see

Presented to the Society, January 18, 1974; received by the editors December 19, 1974.
AMS (MOS) subject classifications (1970). Primary 20D10, 20D25.
Key words and phrases. Finite solvable group, prefrattini subgroups.

[1]). Consequently,

(1.4) $M_{\mathfrak{F}} \subseteq G_{\mathfrak{F}}$ for each subnormal subgroup M in a solvable group G.

It is known that if $G = [A]B$ and A is semisimple with respect to B, then a prefrattini subgroup of B is a prefrattini subgroup of G. Using this fact, (1.4), and an inductive argument, the following is valid:

(1.5) If M is a subgroup that contains $G_{\mathfrak{F}}$ and is normal in a solvable group G, then each prefrattini subgroup of M is contained in a prefrattini subgroup of G.

Using the same fact, (1.3), and an inductive argument, the next conclusions follow:

(1.6) The prefrattini subgroups of a solvable group G are a conjugate class in $G_{\mathfrak{F}}$.

(1.7) A prefrattini subgroup of a solvable group G contains a system normalizer of $G_{\mathfrak{F}}$.

At this point it appears that an understanding of the structure of a prefrattini subgroup is dependent upon the $\mathfrak{F}$-residual. Only partial results are known, such as those above, and the pattern is not clear. Some of these appear in [2], [7], and are generalized to include $\mathfrak{G}$-prefrattini subgroups for saturated formations $\mathfrak{G}$. However (1.6) and (1.7) motivate the concepts introduced in the next section. The result obtained there may indicate the difficulty that lies ahead in trying to formulate precise structure for a prefrattini subgroup.

2. **The prefrattini series.** Let $G_{\mathfrak{F}} = L_0 \supset L_1 \supset \cdots \supset L_n = 1$ denote a lower nilpotent series of length n in $G_{\mathfrak{F}}$. For $\Phi_0 = 1$ and $\Phi_1 = \Phi(G)$, define Φ_{j+1} by $\Phi_{j+1}/L_j^* = \Phi(G/L_j^*)$ for which $L_j^* = \Phi_j L_{n-j}$, $j = 0, \ldots, n$. It can be easily verified that $L_0^* = 1$, $L_{n-1}^* \subset G_{\mathfrak{F}} = \Phi_n = L_0 = L_n^*$, $\Phi_{j+1} \subset L_{j+1}^*$, and $L_j^* \subset \Phi_{j+1}$ for $j = 0, \ldots, n-2$. The series $1 = L_0^* \subseteq \Phi_1 \subset L_1^* \subseteq \Phi_2 \subset \cdots \subset L_{n-1}^* \subset \Phi_n = G_{\mathfrak{F}}$ is a characteristic series of $G_{\mathfrak{F}}$; call this a *prefrattini series*.

A prefrattini subfroup of G covers Φ_j/L_{j-1}^* and avoids L_j^*/Φ_j, for each j.

For a Sylow system $\mathfrak{S}$ of $G_{\mathfrak{F}}$, a lower nilpotent series $\{L_j\}$ of $G_{\mathfrak{F}}$, and for each j, $\mathfrak{S}$ reduces into a Sylow system $\mathfrak{S}_j$ of L_j and into a relative system normalizer $\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j)$ by a result of P. Hall [6]. Call $\{\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j) | j = 0, \ldots, n-2\}$ an $\mathfrak{S}$*-system* of relative system normalizers for $G_{\mathfrak{F}}$. This system is dependent upon $\mathfrak{S}$ and it is not being suggested that there is any arbitrariness to the selection of a Sylow system for the L_j. Also from [6] it is known that the relative system normalizers for each L_j form a class of conjugate subgroups in $G_{\mathfrak{F}}$; all are conjugate under L_j. From this it follows that the $\mathfrak{S}$-systems form a single class of conjugate collections of subgroups in $G_{\mathfrak{F}}$. Because relative system normalizers are preserved under homomorphisms [6] and the known relationships between the Sylow systems of a group and a factor group, it is apparent that if N is a normal subgroup of a solvable group G and $N \subseteq G_{\mathfrak{F}}$, then an $\mathfrak{S}$-system of $G_{\mathfrak{F}}$ is mapped onto an $\mathfrak{S}^*$-system of $G_{\mathfrak{F}}/N$, $\mathfrak{S}^*$ being a Sylow system of $G_{\mathfrak{F}}/N$ into which $\mathfrak{S}$ reduces. A converse also holds.

THEOREM. *For a solvable group G, let $\{\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j) | j = 0, \ldots, n-2\}$ be an $\mathfrak{S}$-system with respect to a prefrattini series $1 = L_0^* \subseteq \Phi_1 \subset L_1^* \subseteq \cdots \subset L_{n-1}^* \subset \Phi_n = G_{\mathfrak{F}}$. Then $I = \bigcap_{j=1}^{n-1} \Phi_j \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1})$ is a prefrattini subgroup of G.*

PROOF. Consider a minimal normal subgroup $N \subseteq \Phi(G) \neq 1$ and the factor group G/N. Then $\{NL_j/N \mid j = 0, \ldots, n\}$ is a lower nilpotent series for $G_{\mathfrak{F}}/N$, $N\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1})/N$ is a relative system normalizer for NL_{n-j-1}/N with respect to the Sylow system $\mathfrak{S}^*$ of $G_{\mathfrak{F}}/N$ into which $\mathfrak{S}$ reduces, and the prefrattini series for $(G/N)_{\mathfrak{F}}$ is $N \subseteq \Phi_1/N = \overline{\Phi}_1 \subset NL_1^*/N = \overline{L}_1^* \subseteq \cdots \subset NL_{n-1}^*/N = \overline{L}_{n-1}^* \subset \Phi_n/N = G_{\mathfrak{F}}/N$. The result is valid whenever $G_{\mathfrak{F}}$ is nilpotent. By induction, $\bigcap_{j=1}^{n-1}\overline{\Phi}_j(N\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1})/N)$ is a prefrattini subgroup W/N of G/N with respect to the Sylow system $\mathfrak{S}N/N$ (see [7]). Since

$$W/N = \bigcap_{j=1}^{n-1} (\Phi_j/N)\big(N\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1})/N\big)$$
$$= \bigcap_{j=1}^{n-1} \big(\Phi_j\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1})/N\big) = \left(\bigcap_{j=1}^{n-1} (\Phi_j\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1}))/N\right),$$

then $W = \bigcap_{j=1}^{n-1}\Phi_j\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1})$.

Suppose that $\Phi(G) = 1$. Since the result is valid for $G_{\mathfrak{F}}$ nilpotent, assume that $n \geqslant 2$. Then $G_{\mathfrak{F}} = [L_{n-1}]M$ for $M = \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2})$ by a result of R. Carter [3]. The semisimplicity of L_{n-1} with respect to M implies that a prefrattini subgroup of M is a prefrattini subgroup of G. Let $\mathfrak{S}$ denote the extension of a Sylow system $\mathfrak{S}^*$ of M to a Sylow system of G and W a prefrattini subgroup of M relative to $\mathfrak{S}^*$. $M_{\mathfrak{F}}$ has nilpotent length $n-1$ and if $\{\overline{L}_j \mid j = 0, \ldots, n-1\}$ denotes a lower nilpotent series for $M_{\mathfrak{F}}$, then $L_j = L_{n-1}\overline{L}_j$ for $j = 0, \ldots, n-1$. Furthermore the prefrattini series $1 = \overline{L}_0^* \subseteq \overline{\Phi}_1 \subset \overline{L}_1^* \subseteq \overline{\Phi}_2 \subset \cdots \subset \overline{L}_{n-2}^* \subset \overline{\Phi}_{n-1} = M_{\mathfrak{F}}$ of $M_{\mathfrak{F}}$ is associated with the prefrattini series of $G_{\mathfrak{F}}$ by $L_j^* = L_{n-1}\overline{L}_{j-1}^*$ and $\Phi_j = L_{n-1}\overline{\Phi}_{j-1}$.

If R_j is a relative system normalizer of $\overline{L}_j$ in M via $\mathfrak{S}^*$, then $R_j \subseteq \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j)$. Moreover

$$R_j = L_{n-1}R_j/L_{n-1} \cong L_{n-1}\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j)/L_{n-1} \cong \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j)/\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j) \cap L_{n-1}.$$

Each $x \in \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_j) \cap L_{n-1}$, for $j \leqslant n-2$, normalizes $\mathfrak{S}_{n-2}$. Hence $x \in L_{n-1} \cap \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2}) = 1$. Therefore $R_j = N_{G_{\mathfrak{F}}}(\mathfrak{S}_j)$.

By induction, $W = \bigcap_{i=1}^{n-2}\Phi_i R_{n-i-2}$ is a prefrattini subgroup of M. Then

$$I = \bigcap_{j=1}^{n-1} \Phi_j\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1}) = \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2}) \cap \left(\bigcap_{j=2}^{n-1} \Phi_j\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1})\right)$$
$$= \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2}) \cap \left(\bigcap_{j=2}^{n-1} L_{n-1}\overline{\Phi}_{j-1}R_{n-j-1}\right)$$
$$= \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2}) \cap L_{n-1}\left(\bigcap_{j=2}^{n-1} \overline{\Phi}_{j-1}R_{n-j-1}\right)$$
$$= \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2}) \cap L_{n-1}\left(\bigcap_{i=1}^{n-2} \overline{\Phi}_i R_{n-i-2}\right)$$
$$= \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2}) \cap L_{n-1}W = W\big(\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-2}) \cap L_{n-1}\big) = W.$$

Therefore I is a prefrattini subgroup of G.

Another proof for (1.7) emerges. Just note that $\mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}) \subseteq \bigcap_{j=1}^{n-1} \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1}) \subseteq I$. Moreover (1.6) also follows from the Theorem. Since the $\mathfrak{S}$-systems form a complete conjugate class, then so does $\{\bigcap_{j=1}^{n-1} \Phi_j \mathbf{N}_{G_{\mathfrak{F}}}(\mathfrak{S}_{n-j-1}) | \mathfrak{S}\}$. In passing, if $\Phi_j = L_{j-1}^*$ for $j = 1, \ldots, n-1$ in a prefrattini series for $G_{\mathfrak{F}}$, then I is a system normalizer of $G_{\mathfrak{F}}$.

ACKNOWLEDGEMENT. I thank the Mathematics Institute at the University of Warwick for the services provided to me while this paper was being prepared.

BIBLIOGRAPHY

1. H. Bechtell, *On the structure of solvable nC-groups*, Rend. Sem. Mat. Univ. Padova **47**(1972), 13–22. MR**48** #2253.

2. B. Brewster, *Prefrattini subgroups of finite solvable groups*, Manuscript, 1974.

3. R. W. Carter, *Splitting properties of soluble groups*, J. London Math. Soc. **36**(1961), 89–94. MR**24** #A3202.

4. W. Gaschütz, *Über die Φ-Untergruppe endlicher Gruppen*, Math. Z. **58**(1953), 160–170. MR**15**, 285.

5. ——— , *Praefrattinigruppen*, Arch. Math. **13**(1962), 418–426. MR**26** #3784.

6. P. Hall, *On the system normalizers of a solvable group*, Proc. London Math. Soc. **43**(1937), 507–528.

7. T. Hawkes, *Analogues of Prefrattini subgroups*, Proc. Internat. Conf. Theory of Groups (Canberra, 1965), Gordon and Breach, New York, 1967, pp. 145–150. MR**36** #279.

8. B. Huppert, *Endliche Gruppen*. I, Die Grundlehren der math. Wissenschaften, Band 134, Springer-Verlag, Berlin and New York, 1967. MR**37** #302.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NEW HAMPSHIRE 03824