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HYPERFINITENESS AND THE HALMOS-ROHLIN THEOREM FOR NONSINGULAR ABELIAN ACTIONS[1]

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ABSTRACT. THEOREM 1. *Let the countable abelian group G act nonsingularly and aperiodically on Lebesgue space (X, μ). Then for each finite subset $A \subset G$ and $\varepsilon > 0$ $\exists$ finite $B \subset G$ and $F \subset X$ with $\{bF\colon b \in B\}$ disjoint and $\mu[(\bigcap_{a \in A} B - a)F] > 1 - \varepsilon$.*

THEOREM 2. *Every nonsingular action of a countable abelian group on a Lebesgue space is hyperfinite.*

1. **Introduction**. The principal results here are a Halmos-Rohlin theorem for nonsingular actions of a countable abelian group on a Lebesgue measure space, and a proof of their hyperfiniteness. The latter fact has relevance for the group-measure space construction of von Neumann algebras. This construction produces algebras of type III precisely when there is no equivalent measure preserved by the action (see [9, Chapter 4.2]).

These results have already been proved for measure-preserving actions. Katznelson and Weiss [4] and Conze [1] proved a Halmos-Rohlin theorem for measure-preserving actions of $\mathbf{Z}^d$, and Krieger [5] extended this to countable abelian groups. Hyperfiniteness was shown in the measure-preserving case by Dye in the second of his pioneering papers [2] and [3]. However, it seems worthwhile to give a simpler proof of hyperfiniteness even in this case. Finally, Veršik [10] has announced a proof of the hyperfiniteness of nonsingular countable abelian actions. However, the only proof of his of which we are aware [11] has serious gaps.

After this paper was completed, we learned from A. Connes that he and W. Krieger have also proved Theorem 2, apparently by somewhat different methods.

2. **The Halmos-Rohlin theorem**. All transformations act on a fixed Lebesgue measure space (X, μ) (see [8] for the properties of such spaces). An invertible measurable transformation of X is called *nonsingular* if both it and its inverse map μ-null sets to μ-null sets. The group of all such transformations is denoted by $\mathfrak{N}(\mu)$.

Let G be a countable abelian group. A *nonsingular action* of G on (X, μ) is a homomorphism $T\colon G \to \mathfrak{N}(\mu)$. We abbreviate $T(g)(x)$ by gx, $T(g)(F)$ by

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gF for subsets F of X, etc. We say G acts *aperiodically* on (X, μ) if the only element of G which has a fixed point is the identity of G. If $B \subset G$ and $F \subset X$, then BF denotes $\bigcap\{bF: b \in B\}$. A subset F of X is called a B *set* if $\{bF: b \in B\}$ is a disjoint collection. If A and B are subsets of G, then $\bigcap_A B$ denotes $\bigcap\{B - a: a \in A\}$. We are now prepared to state the result of this section.

THEOREM 1. *Let the countable abelian group G act nonsingularly and aperiodically on the Lebesgue space (X, μ). Then for each finite subset A of G and each $\varepsilon > 0$, there exists a finite subset B of G and a measurable B set F with* $\mu[(\bigcap_A B)F] > 1 - \varepsilon$.

PROOF. The proof builds on some ideas in [7]. For measure-preserving actions this proof is actually quite simple. Most of our proof is concerned with using averaging arguments to control the size of sets which are easily shown to be small for measure-preserving actions.

First, suppose that the theorem is true for actions of the d-dimensional integers $\mathbf{Z}^d$. If G is a countable abelian group acting on X, and A is a finite subset of G, we can assume without loss that A generates G. Hence for some integer d and finite group H, we have G isomorphic to $\mathbf{Z}^d \oplus H$. Let ζ be the partition of X into orbits of H, and $q: X \to X/\zeta$ be the quotient map. Then ζ is a measurable partition in the sense of Rohlin [8], and X/ζ is a Lebesgue space under the measure $\mu_\zeta(E) = \mu(q^{-1}E)$. The aperiodic action of $\mathbf{Z}^d$ on X induces one on X/ζ. Let A_0 be the projection of A into $\mathbf{Z}^d$. By our initial assumption, for every $\varepsilon > 0$ there is a $B_0 \subset \mathbf{Z}^d$ and $F_0 \subset X/\zeta$ with $\mu_\zeta[(\bigcap_{A_0} B_0)F_0] > 1 - \varepsilon$. Let $F \subset X$ be a measurable cross section of q restricted to $q^{-1}(F_0)$, and $B = B_0 \oplus H$. Then F is a B set, and since $\bigcap_A B = (\bigcap_{A_0} B_0) \oplus H$, we have

$$\mu\left[\left(\bigcap_A B\right)F\right] = \mu_\zeta\left[\left(\bigcap_{A_0} B_0\right)F_0\right] > 1 - \varepsilon.$$

Thus we may assume that $G = \mathbf{Z}^d$. The basic strategy is the following. We begin by showing that for arbitrarily large cubes Q in $\mathbf{Z}^d$ there is a Q set F such that for every t in a cube one fourth the size of Q, the set $(Q + t)F$ fills out at least a fixed proportion of F. The same argument is applied to fill out a fixed proportion of the remainder of X with a smaller cube, and these two constructions are combined to fill out a larger proportion of the entire space using the smaller cube. This combination involves an averaging argument which uses some flexibility in the original choice of Q. Repeating this procedure eventually fills out as much of X as desired.

Let Q_P be the cube $\{0, 1, \ldots, P - 1\}^d$, and

$$R_P = \{-P, -P + 1, \ldots, P - 1\}^d,$$

so that R_P is made up of 2^d translates of Q_P. Let $B_N(Q_P)$ be a barrier of thickness N surrounding Q_P, namely

$$B_N(Q_P) = \{t \in \mathbf{Z}^d: -N \leqslant t_j < P + N\} \setminus Q_P.$$

If L divides P, let $S_L(Q_P)$ be Q_P "shrunk" symmetrically so that a proportion

$1/L$ is removed from its surface, namely $S_L(Q_P) = \{t \in \mathbf{Z}^d: P/L \leqslant t_j < P - P/L\}$.

Say an integer L *works* for the real number β if there is some M such that for all multiples P of LM there exists a Q_P set F with $\mu[S_L(Q_P)F] \geqslant \beta$. Let α be the supremum of the numbers β for which some integer works. We will show that $\alpha = 1$. This will prove the theorem, since for a finite subset A of G, we have $S_L(Q_P) \subset \bigcap_A Q_P$ for sufficiently large P.

We first show that 4 works for 4^{-d}, so that $\alpha > 0$. Aperiodicity of the action guarantees that for every integer P and every subset E of positive measure, there is a Q_P set of positive measure contained in E. Zorn's Lemma provides a maximal Q_P set F; that is, a measurable subset F such that if $F' \supset F$ and F' is also a Q_P set, then $\mu(F' \setminus F) = 0$. We claim that $\mu(R_P F) = 1$. For otherwise, $X \setminus R_P F$ would contain a Q_P set of positive measure, and this could clearly be combined with F to produce a larger Q_P set. Now assume that P is divisible by 4. Then R_P is covered by 4^d translates of $Q_{P/2}$, and so there must be at least one such translate, say $Q_{P/2} + t$, for which $\mu[(Q_{P/2} + t)F] \geqslant 4^{-d}$. Choose t' so that $Q_{P/2} + t - t' = S_4(Q_P)$. Set $F_1 = t'F$. Then F_1 is a Q_P set and $\mu[S_4(Q_P)F_1] = \mu[(Q_{P/2} + t)F] \geqslant 4^{-d}$.

Now suppose that $\beta < \alpha$. Let ξ, η be positive numbers. We will show that $\beta - \xi + 4^{-d}(1 - \beta - \eta)$ is also less than α. Hence $\beta + 4^{-d}(1 - \beta) \leqslant \alpha$. But the only number α in $(0, 1]$ which can have this property for all $\beta < \alpha$ is $\alpha = 1$.

The proof of our assertion is based on the following two statements.

(i) If $\eta > 0$, then for every sufficiently large even integer M, integer L, multiple $P = NLM$ of LM, and Q_P set F, for over $9/10$ of the elements t in $Q_{P/L}$ we have

$$\mu[B_{2N}(Q_P)(tF)] < \eta.$$

(ii) If $\xi > 0$, then for every sufficiently large integer L, integer M, multiple $P = NLM$ of LM by a multiple N of L, and Q_P set F, for over $9/10$ of the t in $Q_{P/L}$ we have

$$\mu[S_L(Q_N)(NQ_{P/N})(tF)] > \mu[Q_P(tF)] - \xi.$$

We will prove statement (i). The proof of (ii) is similar, and we omit it.

Divide $B_{2N}(Q_P)$ into lower dimensional "slabs" as follows. For each nonempty subset Λ of $\{1, 2, \ldots, d\}$ and each $\sigma: \Lambda \to \{-1, 1\}$, let

$$\begin{aligned} B_\Lambda^\sigma = \{t: 0 \leqslant t_j < P \text{ if } j \notin \Lambda, -2N \leqslant t_j < 0 \text{ if } \sigma(j) = -1, \\ P \leqslant t_j < P + 2N \text{ if } \sigma(j) = 1\}. \end{aligned}$$

Then $B_{2N}(Q_P)$ is the disjoint union of the B_Λ^σ. The number of these slabs is easily seen to be $3^d - 1$. Let

$$Q_\Lambda = \{2Nt: t_j = 0 \text{ if } j \notin \Lambda, 0 \leqslant t_j < M/2 \text{ if } j \in \Lambda\}.$$

Since $B_\Lambda^\sigma + Q_\Lambda$ is contained in a translate of Q_P, and since the collection $\{B_\Lambda^\sigma + t: t \in Q_\Lambda\}$ is disjoint, it follows that $B_\Lambda^\sigma F$ is a Q_Λ set.

Let the cardinality of a set A be denoted by $|A|$. Then $Q_{P/L}$ consists of a disjoint union of $|Q_{P/L}|/|Q_\Lambda|$ translates of Q_Λ. Thus

$$\sum_{t \in Q_{P/L}} \mu[B_\Lambda^\sigma(tF)] \leqslant \frac{|Q_{P/L}|}{|Q_\Lambda|}.$$

Since $|Q_\Lambda| \geqslant M/2$, the right-hand side is bounded by $2|Q_{P/L}|/M$. Summing over Λ and σ shows that

$$\sum_{t \in Q_{P/L}} \mu[B_{2N}(Q_P)(tF)] \leqslant \frac{2(3^d - 1)|Q_{P/L}|}{M},$$

and hence that

$$\frac{1}{|Q_{P/L}|}\left|\left\{t \in Q_{P/L}: \mu[B_{2N}(Q_P)(tF)] > \frac{20(3^d - 1)}{M}\right\}\right| < \frac{1}{10}.$$

If $M > 20(3^d - 1)/\eta$, the desired inequality in (i) holds.

We now complete the proof of the theorem using (i) and (ii). Suppose $\beta < \alpha$, and choose L_1 to work for β. This means that there exists an M_1 such that for any multiple P of $L_1 M_1$ there is a Q_P set F with $\mu[S_{L_1}(Q_P)F] \geqslant \beta$. Let M be an even multiple of M_1 and so large that (i) holds. Let L be a multiple of L_1 and so large that (ii) holds. Let P be a multiple of LM by a multiple N of $2L$. Hence for the Q_P set F with $\mu[S_{L_1}(Q_P)F] \geqslant \beta$, there exists some t in $Q_{P/L}$ so that the inequalities in both (i) and (ii) hold. Let $F_1 = tF$, and let

$$E = X \setminus (Q_P \cup B_{2N}(Q_P))F_1.$$

Choose a maximal Q_N set F_2 in E. Then, arguing as before, $E \subset R_N F_2$ except for a null set. Now R_N is the union of 4^d translates of $Q_{N/2}$, so for one of these, say $Q_{N/2} + u$, we must have

$$\mu[\{(Q_{N/2} + u)F_2\} \cap E] \geqslant 4^{-d}\mu(E).$$

Choose u' so that $Q_{N/2} + u - u' = S_4(Q_N)$, and put $F_3 = u'F_2$. Finally, put $F_4 = (NQ_{P/N})F_1$. We will check that $F' = F_3 \cup F_4$ is a Q_N set for which

$$\mu[S_L(Q_N)F'] > \beta - \xi + 4^{-d}(1 - \beta - \eta).$$

This will show that L works for $\beta - \xi + 4^{-d}(1 - \beta - \eta)$, and complete the proof.

The set F_2 was chosen to be a Q_N set, so the same holds for F_3. Since F_1 is a Q_P set, it follows that F_4 is a Q_N set, and

$$Q_N F_4 = (Q_N + NQ_{P/N})F_1 = Q_P F_1.$$

Now $u' \in R_N$, so that

$$Q_N F_3 = Q_N(u'F_2) \subset R_{2N}E.$$

Since $R_{2N}E$ is disjoint from $Q_P F_1$, we have that $F_3 \cup F_4$ is also a Q_N set.

We estimate the measures of the disjoint sets $S_L(Q_N)F_3$ and $S_L(Q_N)F_4$ separately. By (i) we have

$$\begin{aligned}\mu[S_L(Q_N)F_3] &\geqslant \mu[S_4(Q_N)F_3] = \mu[(Q_{N/2} + u)F_2] \geqslant 4^{-d}\mu(E)\\ &\geqslant 4^{-d}(1 - \mu[Q_P(tF)] - \mu[B_{2N}(Q_P)(tF)])\\ &\geqslant 4^{-d}(1 - \mu[Q_P(tF)] - \eta).\end{aligned}$$

Also, using (ii) we have

$$\mu[S_L(Q_N)F_4] = \mu[S_L(Q_N)(NQ_{P/N})(tF)] \geqslant \mu[Q_P(tF)] - \xi.$$

Thus

$$\mu[S_L(Q_N)F'] \geqslant \mu[Q_P(tF)] - \xi + 4^{-d}(1 - \mu[Q_P(tF)] - \eta).$$

Since $t \in Q_{P/L}$, we have $Q_P + t \supset S_L(Q_P)$, so that

$$\mu[Q_P(tF)] \geqslant \mu[S_L(Q_P)F] \geqslant \mu[S_{L_1}(Q_P)F] \geqslant \beta.$$

Applying this to the right side of the previous inequality gives the desired result.

3. **Hyperfiniteness**. A nonsingular action of a countable group G on X is called hyperfinite if for each finite subset A of G and each $\varepsilon > 0$, there exists some finite group $K \subset \mathfrak{N}(\mu)$ such that $Kx \subset Gx$ for almost every x, and such that for each $a \in A$ there is some $k \in K$ with $\mu(\{x\colon ax \neq kx\}) < \varepsilon$.

This definition (in the measure-preserving case) is due to Dye [2]. Two equivalent definitions are the following:

(1) there is some nonsingular action of $\mathbf{Z}$ on X such that $\mathbf{Z}x = Gx$ for almost every x;

(2) there exist finite groups $G_1 \subset G_2 \subset \cdots$ of nonsingular transformations of X with $\bigcup G_n x = Gx$ for almost every x.

The proof of the equivalence of these with the original definition is in [2] and [5].

The first lemma describes the aperiodic decomposition of X.

LEMMA 1. *Let the countable abelian group G act nonsingularly on X. If H is a subgroup of G, let $X_H = \{x\colon gx = x \text{ if and only if } g \in H\}$. Then X is the disjoint union of the X_H, each X_H is measurable and invariant under G, and G/H acts aperiodically on X_H.*

PROOF. Clear.

LEMMA 2. *A nonsingular action of G is hyperfinite if for each finite subset A of G and each $\varepsilon > 0$, there exists a finite subset B of G and a B set F in X such that $\mu[(\bigcap_A B)F] > 1 - \varepsilon$.*

PROOF. Suppose that A is a finite subset of G which contains the identity, and let $\varepsilon > 0$. Choose B and F to satisfy the hypothesis. We construct the required finite group K from B and F as follows. For each permutation π of B, let $T_\pi \in \mathfrak{N}(\mu)$ be defined by

$$T_\pi x = \begin{cases} \pi(b)x & \text{if } x \in bF\ (b \in B), \\ x & \text{if } x \in X \setminus BF. \end{cases}$$

The collection of such T_π forms a finite group K in $\mathfrak{N}(\mu)$. Clearly $Kx \subset Gx$ for every x. If $a \in A$, the map $b \mapsto b + a$ from $\bigcap_A B$ to B extends to a permutation π_a of B. Then $T_{\pi_a} \in K$, and

$$\mu(\{x\colon ax \neq T_{\pi_a}x\}) \geqslant \mu\left[\left(\bigcap_A B\right)F\right] > 1 - \varepsilon.$$

THEOREM 2. *Every nonsingular action of a countable abelian group on a Lebesgue space is hyperfinite.*

PROOF. From the definition of hyperfiniteness, it is clear that it suffices to consider the case where G is finitely generated. Since the number of subgroups of a finitely generated abelian group is countable, by Lemma 1 it suffices to consider aperiodic actions. The result then follows from Theorem 1 and Lemma 2.

REMARK. Lemma 2 gives a criterion for hyperfiniteness of nonsingular actions of countable groups which are not necessarily abelian. Our results show that all countable abelian groups satisfy this criterion. In the measure-preserving case, the conditions on B and F can be replaced by $\mu(BF) > 1 - \varepsilon$ and $|\bigcap_A B| > (1 - \varepsilon)|B|$, the latter being a condition only on the group and not the action.

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