PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

CHARACTERIZATIONS OF URYSOHN-CLOSED SPACES

LARRY L. HERRINGTON

ABSTRACT. This paper gives characterizations of Urysohn-closed and minimal Urysohn spaces, some of which make use of nets.

1. **Introduction**. Our primary interest is the investigation of Urysohn-closed and minimal Urysohn spaces. Characterizations of Urysohn-closed and minimal Urysohn spaces are given in terms of special types of open filterbases [1, p. 101]. Open filterbases, of course, determine nets but not every net determines an open filterbase. We give characterizations of Urysohn-closed and minimal Urysohn spaces in terms of nets and arbitrary filterbases. These characterizations are obtained mainly through the introduction of a type of convergence for filterbases and nets that we call u-convergence.

Throughout, $\mathrm{cl}(A)$ will denote the closure of a set A.

2. **Preliminary definitions and theorems**. Let X be a topological space and let G and H be open sets in X containing a point $p \in X$. Then G and H will be called an ordered pair of open sets containing p (denoted by (G, H)) if $p \in G \subset \mathrm{cl}(G) \subset H$.

DEFINITION 2.1. Let X be a topological space and let $\mathfrak{F} = \{A_\alpha: \alpha \in \Delta\}$ be a filterbase in X. Then $\mathfrak{F}$ u-converges to $x \in X$ ($\mathfrak{F} \to_u x$) if for each ordered pair of open sets (G, H) containing x there exists an $A_\alpha \in \mathfrak{F}$ such that $A_\alpha \subset \mathrm{cl}(H)$. The filterbase $\mathfrak{F}$ u-accumulates to $x \in X$ ($\mathfrak{F} \propto_u x$) if for each ordered pair of open sets (G, H) containing x and for each $A_\alpha \in \mathfrak{F}$, $A_\alpha \cap \mathrm{cl}(H) \neq \emptyset$.

Convergence and accumulation of filterbases in the usual sense, of course, imply u-convergence and u-accumulation, respectively. However, the converses do not hold as the next example shows.

EXAMPLE 2.2. Let $I = [0, 1]$ have as a subbase the usual open sets together with the set $A = \{r: 1/4 < r < 3/4$ and r is rational$\}$. Let the filterbase $\mathfrak{F}$ consist of a single element $B = \{x: 1/3 < x < 2/3$ and x is irrational$\}$ and let $x = 1/2$. The filterbase $\mathfrak{F}$ does not converge or accumulate in the usual sense to x but $\mathfrak{F} \propto_u x$.

There are a number of theorems concerning u-convergence and u-accumulation whose statements parallel those of convergence and accumulation in the usual sense. We give a sample of some of these theorems but omit their straightforward proofs.

Received by the editors October 28, 1974.

AMS (MOS) subject classifications (1970). Primary 54D30.

Key words and phrases. Urysohn-closed and minimal Urysohn spaces.

THEOREM 2.3. *In a topological space X the following properties hold*:

(a) *If $\mathfrak{F}$ is a filterbase in X such that $\mathfrak{F}$ u-converges to $x \in X$, then $\mathfrak{F}$ u-accumulates to x. If X is a Urysohn space and if $\mathfrak{F}$ converges to $x \in X$, then $\mathfrak{F}$ u-accumulates at no point other than x.*

(b) *Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ be two filterbases in X where $\mathfrak{F}_2$ is stronger than $\mathfrak{F}_1$. Then $\mathfrak{F}_1$ u-accumulates to $x \in X$ if $\mathfrak{F}_2$ u-accumulates to x.*

(c) *A filterbase $\mathfrak{F}_1$ u-accumulates to $x \in X$ if and only if there exists a filterbase $\mathfrak{F}_2$ stronger than $\mathfrak{F}_1$ such that $\mathfrak{F}_2$ u-converges to x.*

(d) *A maximal filterbase $\mathfrak{M}$ in X u-accumulates to $x \in X$ if and only if $\mathfrak{M}$ u-converges to x.*

DEFINITION 2.4. Let X be a topological space and let $\mathcal{O} : D \to X$ be a net in X. Then $\mathcal{O}$ u-converges to $x \in X$ ($\mathcal{O} \to_u x$) if for each ordered pair of open sets (G, H) containing x, there exists a $b \in D$ such that $\mathcal{O}(T_b) \subset \mathrm{cl}(H)$ (where $T_b = \{c \in D: b \prec c\}$). The net $\mathcal{O}$ u-accumulates to $x \in X$ ($\mathcal{O} \propto_u x$) if for each ordered pair of open sets (G, H) containing x and for every $b \in D$, $\mathcal{O}(T_b) \cap \mathrm{cl}(H) \neq \varnothing$.

Of course, if $\mathcal{O} : D \to X$ is a net in X, the family $\mathfrak{F}(\mathcal{O}) = \{\mathcal{O}(T_b): b \in D\}$ is a filterbase in X and it is routine to verify that:

(a) $\mathfrak{F}(\mathcal{O}) \to_u x \in X$ if and only if $\mathcal{O} \to_u x$.

(b) $\mathfrak{F}(\mathcal{O}) \propto_u x \in X$ if and only if $\mathcal{O} \propto_u x$.

Conversely, every filterbase $\mathfrak{F}$ in X determines a net $\mathcal{O} : D \to X$ such that:

(a) $\mathfrak{F} \to_u x \in X$ if and only if $\mathcal{O} \to_u x$.

(b) $\mathfrak{F} \propto_u x \in X$ if and only if $\mathcal{O} \propto_u x$.

The construction of such a net is the same as that of [2, p. 213].

We next state a few theorems concerning u-convergence for nets.

THEOREM 2.5. *In a topological space X the following properties hold*:

(a) *If $\mathcal{O}$ is a net in X such that $\mathcal{O}$ u-converges to $x \in X$, then $\mathcal{O}$ u-accumulates to x. If X is a Urysohn space and if $\mathcal{O}$ converges to $x \in X$, then $\mathcal{O}$ u-accumulates at no point other than x.*

(b) *A net $\mathcal{O}$ u-accumulates to $x \in X$ if and only if there exists a subnet of $\mathcal{O}$ u-converging to x.*

(c) *A universal net $\mathcal{O}$ u-accumulates to $x \in X$ if and only if $\mathcal{O}$ u-converges to x.*

3. **Filterbases and net characterizations of Urhysohn-closed spaces.** An open filterbase base $\mathfrak{F}$ in X is a Urysohn filterbase if and only if for each $p \notin A(\mathfrak{F})$ (where $A(\mathfrak{F})$ denotes the set of accumulation points of $\mathfrak{F}$), there is an open neighborhood U of p and some $V \in \mathfrak{F}$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(V) = \varnothing$ [3]. An open cover $\mathfrak{U}$ of a space X is a Urysohn open cover if there exists an open cover $\mathfrak{V}$ of X with the property that for each $V \in \mathfrak{V}$, there is a $U \in \mathfrak{U}$ such that $\mathrm{cl}(V) \subset U$. A Urysohn space X is Urysohn-closed provided X is a closed set in every Urysohn space in which it can be embedded [1].

LEMMA 3.1. *Let $\mathfrak{F} = \{O_\alpha: \alpha \in \Delta\}$ be an open Urysohn filterbase on X. Then $A(\mathfrak{F}) = A_u(\mathfrak{F})$ (where $A_u(\mathfrak{F})$ denotes the set of u-accumulation points of $\mathfrak{F}$).*

PROOF. Clearly we only need to show that $A_u(\mathfrak{F}) \subset A(\mathfrak{F})$. Suppose $p \notin A(\mathfrak{F})$. Then there exists an open set U containing p and some $O_\alpha \in \mathfrak{F}$ such that $\mathrm{cl}(U) \cap \mathrm{cl}(O_\alpha) = \varnothing$. The open sets $V = X - \mathrm{cl}(O_\alpha)$ and U form an ordered pair of open sets, (U, V), containing p and have the property that $O_\alpha \cap \mathrm{cl}(V) = \varnothing$. Consequently, $p \not\in A_u(\mathfrak{F})$. Therefore, we conclude that $A(\mathfrak{F}) = A_u(\mathfrak{F})$.

Theorem 4.1 of [1, p. 101] gives several characterizations of Urysohn-closed spaces. We offer the following characterizations.

THEOREM 3.2. *Let X be a Urysohn space. Then the following are equivalent*:
(a) *X is Urysohn-closed.*
(b) *Each filterbase $\mathfrak{F}$ in X u-accumulates to some point $x \in X$.*
(c) *Each maximal filterbase $\mathfrak{M}$ in X u-converges to some point $x \in X$.*

PROOF. (a) implies (b). Suppose there exists a filterbase $\mathfrak{F} = \{A_\alpha \in \Delta\}$ in X that does not u-accumulate in X. Then for each $x \in X$ there exists an ordered pair of open sets $(U(x), V(x))$ containing x and some $A_{\alpha(x)} \in \mathfrak{F}$ such that $A_{\alpha(x)} \cap \mathrm{cl}(V(x)) = \varnothing$. Now $\{V(x)\colon x \in X\}$ is a Urysohn open cover of X. Thus by Theorem 4.1 of [1, p. 101], there exists a finite subcollection $\{V(x_i)\colon i = 1, 2, 3, \ldots, n\}$ such that $\bigcup_{i=1}^{n} \mathrm{cl}(V(x_i)) = X$. Since $\mathfrak{F}$ is a filterbase, there exists an $A_{\alpha_0} \in \mathfrak{F}$ such that $A_{\alpha_0} \subset \bigcap_{i=1}^{n} A_{\alpha(x_i)}$, and $A_{\alpha_0} \neq \varnothing$ implies that for some j, $1 \leqslant j \leqslant n$, $A_{\alpha_0} \cap \mathrm{cl}(V(x_j)) \neq \varnothing$. Therefore $A_{\alpha(x_j)} \cap \mathrm{cl}(V(x_j)) \neq \varnothing$ which is a contradiction.

(b) implies (a). Let $\mathfrak{F} = \{O_\alpha\colon \alpha \in \Delta\}$ be an open Urysohn filterbase on X. By Lemma 3.1 and hypothesis (b) we have that $A_u(\mathfrak{F}) = A(\mathfrak{F}) \neq \varnothing$. Therefore X is Urysohn-closed according to Theorem 4.1 of [1, p. 101].

(b) implies (c). Let $\mathfrak{M}$ be a maximal filterbase in X. Then $\mathfrak{M}$ u-accumulates to some point in X by (b) and hence u-converges to that point by Theorem 2.3(d).

(c) implies (b). Let $\mathfrak{F}$ be a filterbase in X. Then there exists a maximal filterbase $\mathfrak{M}$ in X which is stronger than $\mathfrak{F}$. Since $\mathfrak{M}$ u-converges to some point $x \in X$, $\mathfrak{F}$ u-accumulates to x according to Theorem 2.3.

Our discussion in the previous section showed that filterbases and nets are "equivalent" in the sense of u-convergence and u-accumulation. Thus we can now characterize Urysohn-closed spaces in terms of nets.

THEOREM 3.3. *In a Urysohn space X the following are equivalent*:
(a) *X is Urysohn-closed.*
(b) *Each net in X has a u-accumulation point.*
(c) *Each universal net u-converges.*

REMARK 3.4. For each topological space (X, τ) there is a corresponding topological space (X, τ_*) called the semiregular space associated with (X, τ) [1, p. 96]. The topology τ_* is generated by the regular open sets in (X, τ). For each open set U in (X, τ), $\mathrm{cl}(U) = \mathrm{cl}_*(U)$ (where $\mathrm{cl}_*(U)$ denotes the closure of U in (X, τ_*)). Consequently, it follows that a space (X, τ) is Urysohn if and only if (X, τ_*) is Urysohn. Also, it is easy to see that a filterbase $\mathfrak{F}$ on X u-accumulates to x in (X, τ) if and only if $\mathfrak{F}$ u-accumulates to x in (X, τ_*). With this in consideration we give the following theorem.

THEOREM 3.5. *A space (X, τ) is Urysohn-closed if and only if (X, τ_*) is Urysohn-closed.*

PROOF. The result follows from Theorem 3.2 and Remark 3.4.

Theorem 4.2 of [1, p. 101] characterizes minimal Urysohn spaces in terms of open Urysohn filterbases. In terms of arbitrary filterbases and u-convergence, we give the following characterization of minimal Urysohn spaces.

THEOREM 3.6. *Let (X, τ_0) be a Urysohn space. Then X is minimal Urysohn if and only if each filterbase in X possessing at most one u-accumulation point is convergent.*

PROOF. Suppose the condition is given and let $\mathfrak{F}$ be an open Urysohn filterbase on X possessing at most one accumulation point. By Lemma 3.1, $\mathfrak{F}$ possesses at most one u-accumulation point. Consequently, by hypothesis, $\mathfrak{F}$ converges. This shows that X is minimal Urysohn according to Theorem 4.2 of [1, p. 101].

Conversely, assume that X is a minimal Urysohn space and suppose that $\mathfrak{F}_0 = \{A_\alpha: \alpha \in \Delta\}$ is a filterbase on X possessing at most one u-accumulation point. Let $\mathfrak{F}$ be the filter generated by the filterbase $\mathfrak{F}_0$. Since X is Urysohn-closed (see Theorem 4.3(a) of [1, p. 101]), $\mathfrak{F}_0$ has a unique u-accumulation point $x \in X$. It follows that the collection of open sets $\tau_1 = \{U \in \tau_0: U \subset X - \{x\}\} \cup \{V \in \tau_0: V \in \mathfrak{F}\}$ forms a Urysohn topology on X with the property that $\tau_1 \subset \tau_0$. Suppose there is an open set $G(x) \in \tau_0$ containing x such that for each $A_\alpha \in \mathfrak{F}_0$, $A_\alpha \not\subset G(x)$. Then for each open $U(x) \in \tau_1$ containing x, $U(x) \not\subset G(x)$ which shows that $\tau_1 \neq \tau_0$. Therefore (X, τ_0) is not minimal Urysohn, which is a contradiction. We conclude that $\mathfrak{F}_0$ converges to x.

COROLLARY 3.7. *Let X be a Urysohn space. Then X is minimal Urysohn if and only if each net in X possessing at most one u-accumulation point is convergent.*

4. **First countable Urysohn spaces**. A space (X, τ) is called first countable and minimal Urysohn if τ is first countable and Urysohn, and if no first countable topology on X which is strictly weaker than τ is Urysohn. (X, τ) is first countable and Urysohn-closed if τ is first countable and Urysohn, and (X, τ) is a closed subspace of every first countable Urysohn space in which it can be embedded.

THEOREM 4.1. *A first countable Urysohn space X is first countable and Urysohn-closed if each countable filterbase on X u-accumulates to some point $p \in X$.*

PROOF. Let $\mathfrak{F}$ be a countable open Urysohn filterbase on X. By Lemma 3.1, $A(\mathfrak{F}) = A_u(\mathfrak{F}) \neq \varnothing$ which implies that X is first countable and Urysohn-closed according to Theorem 6.3 of [1, p. 107].

THEOREM 4.2. *A first countable Urysohn space X is first countable and Urysohn-closed if each sequence in X u-accumulates to some point $p \in X$.*

PROOF. Suppose that X is not Urysohn-closed. Then there exists a first countable Urysohn space Y and a homeomorphism $h: X \to h(X) \subset Y$ such that $h(X)$ is not closed in Y. Thus there exists a point $p \in Y - h(X)$ (where $p \in \mathrm{cl}(h(X))$) and a sequence, $f: N \to h(X)$, in $h(X)$ converging to p. Since

$h(X)$ is homeomorphic to X, the sequence f u-accumulates to some point $z \in h(X)$. Therefore $z = p$ according to Theorem 2.5(a), which is a contradiction.

We say that a point $p \in X$ is a u-cluster point of $K \subset X$ if for every ordered pair of open sets (G, H) containing p, $\mathrm{cl}(H) \cap (K - \{p\}) \neq \varnothing$. We note that in a Urysohn space X, a point $p \in X$ is a u-cluster point of $K \subset X$ if and only if for each ordered pair of open sets (G, H) containing p, the closure of H contains infinitely many points of K.

LEMMA 4.3. *In a topological space X the following are equivalent*:
(a) *Every countably infinite subset of Y has at least one u-cluster point.*
(b) *Every sequence in X has a u-accumulation point.*

THEOREM 4.4. *A first countable Urysohn space X is first countable and Urysohn-closed if every countably infinite subset of Y has at least one u-cluster point.*

PROOF. The result follows from Theorem 4.2 and Lemma 4.3.

Theorem 6.3 of [1, p. 107] shows that a first countable Urysohn space X is first countable and minimal Urysohn if every countable open Urysohn filterbase on X with a unique accumulation point is convergent. We show (after Lemma 4.5) that a space X is first countable and minimal Urysohn if each sequence in X with a unique u-accumulation point is convergent.

LEMMA 4.5. *If a Urysohn space X has the property that every sequence in X with a unique u-accumulation point is convergent, then every sequence in X has a u-accumulation point.*

PROOF. Suppose (x_n) is a sequence in X with no u-accumulation point. Fix $p \in X$ and define a sequence, (z_n), by $z_n = p$ if n is odd and $z_n = x_{n/2}$ if n is even. It is clear that p is the unique u-accumulation point of (z_n) and that (z_n) does not converge to p.

THEOREM 4.6. *A first countable Urysohn space (X, τ) is first countable and minimal Urysohn if every sequence in X with a unique u-accumulation point is convergent.*

PROOF. Suppose that h: $(X, \tau) \to (Y, \sigma)$ is a bijective continuous mapping onto a first countable Urysohn space (Y, σ). We need to show that h^{-1} is continuous. Let (y_n) be a sequence in Y converging to $y \in Y$. The continuity of h shows that the sequence, $(h^{-1}(y_n))$, has the unique u-accumulation point $h^{-1}(y)$. By hypothesis, $(h^{-1}(y_n))$ converges to $h^{-1}(y)$ showing that h^{-1} is continuous.

REFERENCES

1. M. P. Berri, J. R. Porter and R. M. Stephenson, Jr., *A survey of minimal topological spaces*, General Topology and its Relations to Modern Analysis and Algebra, III (Proc. Conf. Kanpur, 1968), Academia, Prague, 1971, pp. 93–114. MR **43** #3985.

2. J. Dugundji, *Topology*, Allyn and Bacon, Boston, Mass., 1968. MR **33** #1824.

3. C. T. Scarborough, *Minimal Urysohn spaces*, Pacific J. Math. **27** (1968), 611–617. MR **38** #6530.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS, PINE BLUFF, ARKANSAS 71601