QUASI-UNMIXEDNESS AND INTEGRAL CLOSURE OF REES RINGS

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ABSTRACT. For certain Rees rings $R$ of a local domain $R$, the quasi-unmixedness of $R$ is characterized in terms of a certain transform of $R$ being contained in the integral closure of $R$.

1. Introduction. In this paper, a ring shall be a commutative ring with identity. The terminology is basically that of [2] and [12].

Relations between quasi-unmixedness and integral extensions are well known (e.g., [1], [5] and [7]). Also, the study of properties of a ring $R$ via transition to a Rees ring $R = R(R, A)$ of $R$ (conditions on the ideal $A$ depending on the particular discussion) has often been useful. In particular, characterizations of the quasi-unmixedness of $R$ are given in [10] in terms of localizations of $R$ containing $R$ as a quasi-subspace. The $R$-algebra $T = T(uR)$ (Definition 1) is used in [8] to characterize unmixed local domains. Here, equivalences to the quasi-unmixedness of $R$ are given in terms of $T$ being contained in the integral closure of $R$ (Theorem 2).

2. Preliminary concepts. Let $B = (b_1, \ldots, b_k)R$ be an ideal in a Noetherian ring $R$. Let $t$ be an indeterminant, and let $u = 1/t$. The Rees ring $R = R(R, B)$ of $R$ with respect to $B$ is the ring $R = R[u, tb_1, \ldots , tb_k]$. $R$ is a graded Noetherian subring of $R[u, t]$. If $(R, M)$ is a local ring, then $R = (M, u, tb_1, \ldots , tb_k)$ is the unique maximal homogeneous ideal of $R$. Similar to [12, Theorem 11, p. 157], $R$ is a graded subring of $K[u, t]$, where $K$ is the total quotient ring of $R$. (Throughout, $S'$ will denote the integral closure of ring $S$.)

For an ideal $B$ in a ring $R$, the integral closure of $B$ in $R$, denoted $B'$, is the set of all elements in $R$ satisfying an equation of the form $x^n + b_1 x^{n-1} + \cdots + b_n = 0$, where $b_i \in B^i$, $i = 1, \ldots, n$. It is known [4, p. 523] that $B'$ is an ideal in $R$. In particular, if $B = bR$ is a regular principal ideal, then $(bR)_a = \{r \in R; r/b \in R'\} = bR' \cap R$ [6, Lemma 1].

Definition 1. Let $b$ be a regular nonunit in a ring $R$. Define $T(bR) = \{c_k/b^k; c_k \in (b^k R)^{(1)}\}$, for all large $k$, where $(b^k R)^{(1)}$ is the set of elements of $R$ that are in each height one primary component of $b^k R$.

Remark. The following are shown in [11].

(1) $T(bR)$ is contained in $R'$ if and only if each height one prime divisor of

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$bR'$ contracts to a height one prime (divisor of $bR$) in $R$.

(2) $b^nT(bR)$ is a finite intersection of height one primary ideals. Also $b^nT(bR) \cap R = (b^nR)^{(1)}$.

(3) Define $R^{(1)} = \cap \{R(P); P$ is a height one prime divisor of a principal ideal generated by a nonzero divisor in $R\}$, where $(P)$ denotes the set of regular elements in $R - P$. Then $T(bR) = R[1/b] \cap R^{(1)}$.

3. Characterizations of quasi-unmixed local domains. Several preliminary results on completions are given to show that the condition $T \subseteq R'$ is equivalent to a similar condition for the completion $R^*$ of $R$ (Corollary 1). This is used to give equivalences to the quasi-unmixedness of a local domain (Theorem 2).

**Lemma 1.** Let $B$ be an $M$-primary ideal of a local ring $(R, M)$. Let $R = R(R, B)$. Let $p$ be a prime ideal of $R$ with $uR \subseteq p$. Then $(M, u)R \subseteq p$, and so all prime ideals containing $uR$ lie over $M$.

**Proof.** Since $u$ is in $p$, $B = uR \cap R \subseteq p \cap R$. But $B$ is $M$-primary, so $M \subseteq p \cap R$, i.e., $M = p \cap R$. Q.E.D.

**Lemma 2.** Let $R$ be as in Lemma 1 and $S = R(R^*, BR^*)$. Let $R$ (resp., $R'$) be the maximal homogeneous ideal of $R$ (resp., $S$), and let $R^*$ (resp., $S^*$) be the completion of $R$ (resp., $S$) with respect to the $M$ (resp., $M'$)-adic topology. Then $R^* = S^*$ is the completion $(R_{R'})^* = (S_{S'})^*$ of $R_{R'}$ and $S_{S'}$.

**Proof.** $R_{R'}$ is a dense subspace of $S_{S'}$ [8, Lemma 3.2] and $R^*$ (resp., $S^*$) is the natural completion of $R_{R'}$ (resp., $S_{S'}$) [3, Theorem 32, p. 434]. Q.E.D.

**Lemma 3.** Let $R$, $R^*$, $B$, $R$ and $S$ be as in Lemma 2. Also, assume that $B$ is generated by a system of parameters. Let $T = T(uR)$ and $T^* = T(uS)$. Then $N = (M, u)R((M, u)R) \cap T$ (resp., $N^* = (M^*, u)R((M^*, u)R) \cap T^*$) is the only prime divisor of $uR$ (resp., $uS^*$).

**Proof.** By [8, Remark 3.10(ii)], $(M, u)R$ is the only height one prime divisor of $uR$. By the one-to-one correspondence (and denseness) in [8, Lemma 3.2], $(M^*, u)S = (M, u)S^*$ is the only height one prime divisor of $uS$, and by the one-to-one correspondence in [11, Lemma 2(9)], $N$ (resp., $N^*$) is the only height one prime divisor of $uT$ (resp., $uT^*$). By Remark (2), this ideal has no imbedded prime divisors. Q.E.D.

**Theorem 1.** With the notation of Lemma 2, let $p \subseteq P$ be an inclusion of prime ideals in $R$ with $u \in p$. Then the following statements hold:

(1) $R/p$ is a locally unmixed, pseudo-geometric domain [2, p. 131].

(2) $pR_p^*$ is a semiprime, unmixed ideal in the completion $R_p^*$ of $R^*_p$.

(3) In the completion $R_p^*$ of $R_p$, $pR_p^*$ has pure height equal to height $p$ and has pure depth equal to depth $pR_p^*$.

(4) $pR_p^* = pS^*$ has pure height equal to height $p$, where $p$ is contained in the maximal homogeneous ideal of $R$.

**Proof.** Since $p \cap R = M$ (Lemma 1), $R/p = R/M[u^*, (tB)^*]$, where $X^*$ denotes $X$ modulo $p$. Thus $R/p$ is finitely generated as a ring over the field...
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$R/M$, and so is locally unmixed [2, (34.9)], and pseudo-geometric [2, (36.5)].
This shows (1). By localizing to $R_p$, (2) follows from [2, (36.4)] and (1).

For (3), since $pR_p$ is an unmixed ideal (by (2)), it has pure depth equal to
depth $pR_p = depth pR$. Since $pR_p$ is semiprime, that it has pure height
equal to height $p$ follows from [2, (22.9)]. (4) is a special case of (3) since
$R^*_R = R^* = S^*$ by Lemma 2. Q.E.D.

**Corollary 1.** Let the notation be as in Lemma 2. Then $\mathcal{I}(uR) \subseteq R'$ if and
only if $\mathcal{I}(uS) \subseteq S'$.

**Proof.** Since $(u^nR)_a = u^nR \cap R$ and $R'$ and $R$ are graded subrings of
$K[u, t]$, it follows that $(u^nR)_a$ is a homogeneous ideal in $R$. Therefore, every
prime divisor of $(u^nR)_a$, for $n \geq 1$, and every prime divisor of the homo-
geous ideal $uR$ is contained in the maximal homogeneous ideal $\mathfrak{m}$ of $R$. By
[11, Lemma 4(2)], $\mathcal{I}(uR_{R\mathfrak{m}}) \subseteq R_{R\mathfrak{m}}$ if and only if $\mathcal{I}(uR) \subseteq R'$. Now, let $P$
be a height one prime divisor of $uR_{R\mathfrak{m}}$, and $p = P \cap R$. Then $P_{R\mathfrak{m}} = P^*$
has pure height one (Theorem 1(4)). Therefore, by [11, Corollary 2], $\mathcal{I}(uR_{R\mathfrak{m}})$
$\subseteq R^*_R$ if and only if $\mathcal{I}(uR^*_R) \subseteq R^*_R$. But $R^*_R = (S^*_R)^*$ so the last inclusion
is equivalent to $\mathcal{I}(uS_{R\mathfrak{m}})^* \subseteq (S^*_R)^*$. As above, this is equivalent to $\mathcal{I}(uS_{R\mathfrak{m}})$
$\subseteq (S^*_R)^*$, which, again as above, is equivalent to $\mathcal{I}(uS) \subseteq S'$. Q.E.D.

**Lemma 4.** Let $b$ be a regular nonunit in a Noetherian ring $R$ and $q$ a minimal
prime divisor of zero in $R'$. Then there exists a height one prime divisor $P$ of $bR'$
that contains $q$.

**Proof.** In $R'$, let $Z = \text{rad } (0) = \bigcap_{i=1}^n q_i (q_1 = q)$. Since $Z \subseteq bR'$ [9,
Lemma 2.4], we may pass to $R'/Z = \overline{R}$. $\overline{R}$ is the direct sum of Krull domains
$\oplus_{i=1}^n R'/q_i = \oplus_{i=1}^n \overline{R}e_i$, where the $e_i$ are the associated orthogonal idempo-
tents. A height one prime divisor $p_1$ of $be_1$ in $\overline{R}e_1$ gives rise to the desired $P$.
Q.E.D.

**Theorem 2 (cf. [8, Theorem 5.17]).** Let $(R, M)$ be a local domain of altitude
$n \geq 1$. Then the following statements are equivalent:

(1) $R$ is quasi-unmixed.
(2) For every finitely generated domain $A$ over $R$, and for each multiplicatively
closed subset $S$ of $A$, $(A_S)^{(1)} \subseteq A_S'$.
(3) For every ideal $B$ in $R$, $\mathcal{I}(uR) \subseteq R'$, where $R = R(R, B)$.
(4) There exists an $M$-primary ideal $B$ in $R$ that is generated by a system of
parameters such that $\mathcal{I}(uR) \subseteq R'$, where $R = R(R, B)$.

**Proof.** (1 $\Rightarrow$ 2). By [11, Lemma 1(3) and (5)], it is sufficient to show
$A^{(1)} \subseteq A'$. By [5, Corollary 2.5], $A$ is locally quasi-unmixed. Then, by [7,
Theorem 3.8], each height one prime ideal in $A'$ contracts to a height one
prime in $A$. Thus, by [8, Corollary 5.7], $A^{(1)} \subseteq A'$.
(2 $\Rightarrow$ 3). Since $R$ is a finite extension of $R$, $R^{(1)} \subseteq R'$, by hypothesis. And,
$\mathcal{I}(uR) \subseteq R^{(1)}$.
(3 $\Rightarrow$ 4) is obvious.
(4 $\Rightarrow$ 1). Let $B$ be an $M$-primary ideal of $R$ generated by a system of
parameters. Let $S = R^*(R, BR^*)$ and $\mathcal{I}^* = \mathcal{I}(uS)$ ($R^*$ is the completion of
$S = R(R^*, BR^*)$ and $\mathcal{I}^* = \mathcal{I}(uS)$ ($R^*$ is the completion of
Let $q$ be a minimal prime divisor of zero in $S$. Let $q'$ be the minimal prime divisor of zero in $S'$ that lies over $q$ ($S$ and $S'$ have the same total quotient ring). By Lemma 4, there exists a height one prime divisor $p'$ of $uS'$ that contains $q'$. By Remark 1, $p' \cap S = p$ is a height one prime divisor of $uS$. Hence, $q \subseteq p = (M^*, u)S$ (Lemma 3). Since $q$ was an arbitrary minimal prime divisor of zero in $S$, $R$ is quasi-unmixed [10, Corollary 9]. Q.E.D.

By combining Theorem 2 and the Remark, further characterizations of the quasi-unmixedness of $R$ can be obtained.

**Bibliography**