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GROUP RINGS WITH SIMPLE AUGMENTATION IDEALS

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ABSTRACT. Group algebras of algebraically closed groups and of universal groups are shown to have simple augmentation ideals and to be primitive.

In recent years a large number of examples of primitive group rings have been constructed. In this note we offer some additional examples. However here primitivity is really a secondary consideration since it follows from the even more surprising property that in these group rings $K[G]$ the augmentation ideal $\omega(K[G])$ is the unique proper ideal. The following theorem has a rather unwieldly hypothesis. Nevertheless, as will be apparent, it is precisely what is needed to handle the families of algebraically closed groups and universal groups.

THEOREM. *Let G be a simple group, let K be a field and let q be a prime different from the characteristic of K. Suppose that for any finite number of distinct elements $1 = x_0, x_1, \ldots, x_n \in G$ there exist elements $y_0, y_1, \ldots, y_n \in G$ such that*

(1) $\langle y_i^{x_j} | i = 0, 1, \ldots, n, j = 0, 1, \ldots, n\rangle$ is an elementary abelian q-group.

(2) $\langle (y_i, x_i) | i = 1, 2, \ldots, n\rangle$ has order precisely q^n.

(3) $\langle y_0^{x_j} | j = 0, 1, \ldots, n\rangle$ has order precisely q^{n+1}.

Then $\omega(K[G])$ is the unique proper ideal of $K[G]$. Furthermore $K[G]$ is primitive.

PROOF. Suppose that I is a nonzero proper ideal of $K[G]$. We proceed in a series of steps.

Step 1. Let $\alpha \in I$, $\alpha \neq 0$. Then we can assume that 1 occurs in the support of α and we write $\alpha = \sum_0^n k_i x_i^{-1}$ with $1 = x_0, x_1, \ldots, x_n$ distinct elements of G and with $k_0 \neq 0$. We apply the hypothesis of this theorem to these elements and let $y_1, y_2, \ldots, y_n$ be given as in (1) and (2). Thus by (1) if A is the group $A = \langle y_i^{x_j} | i = 1, 2, \ldots, n, j = 0, 1, \ldots, n\rangle$ then A is an elementary abelian q-group. Set $z_i = (y_i, x_i) = y_i^{-1} x_i^{-1} y_i x_i$ for $i = 1, 2, \ldots, n$.

We show now by inverse induction on s with $n \geqslant s \geqslant 0$ that I contains an element

$$\beta_s = \sum_{i=0}^{s} \beta_{si} x_i^{-1}$$

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with $\beta_{si} \in K[A]$ and such that for $s < n$

$$\beta_{s0} = k_0(z_n - 1)(z_{n-1} - 1)\cdots(z_{s+1} - 1).$$

First for $x = n$ we merely take $\beta_s = \alpha$. Now suppose we have β_s as above contained in I with $s > 0$. Then $y_s^{-1}\beta_s y_s$ and $z_s\beta_s$ both belong to I and hence

$$\beta_{s-1} = z_s\beta_s - y_s^{-1}\beta_s y_s \in I.$$

Furthermore since A is abelian and $y_s, z_s \in A$ we have

$$\begin{aligned}\beta_{s-1} &= \sum_{i=0}^{s} z_s\beta_{si}x_i^{-1} - \sum_{i=0}^{s} y_s^{-1}\beta_{si}x_i^{-1}y_s \\ &= \sum_{i=0}^{s} \beta_{si}\{z_s - (y_s, s_i)\}x_i^{-1} \\ &= \sum_{i=0}^{s-1} \beta_{si}\{z_s - (y_s, x_i)\}x_i^{-1}\end{aligned}$$

since $z_s = (y_s, x_s)$. Thus since $x_0 = 1$, $\beta_{s-1,0}$ has the appropriate form and the induction step is proved.

In particular, when $s = 0$ we conclude that

$$\beta_0 = k_0(z_n - 1)(z_{n-1} - 1)\cdots(z_1 - 1) \in I.$$

Furthermore $k_0 \neq 0$ and

$$\langle z_1, z_2, \ldots, z_n\rangle = \langle z_1\rangle \times \langle z_2\rangle \times \cdots \times \langle z_n\rangle$$

is a direct product of cyclic groups of order q by property (2). Hence $\beta_0 \neq 0$ and we have shown that there exists a finite elementary abelian q-subgroup A of G with $I \cap K[A] \neq 0$.

Step 2. Let A be a finite elementary abelian q-subgroup of G, whose existence is guaranteed by Step 1, such that $I \cap K[A] \neq 0$. Write $A = \{1 = x_0, x_1, \ldots, x_n\}$ and let $y_0 \in G$ be given satisfying (1) and (3). Thus by (1), if B is the group $B = \langle y_0^{x_j} | j = 0, 1, \ldots, n\rangle$, then B is an elementary abelian q-group normalized by A. Furthermore since $|B| = q^{n+1}$ and $n + 1 = |A|$, it is clear that A acts faithfully on B so $A \cap B = \langle 1\rangle$. Thus we conclude first that $H = BA$ is the semidirect product of B by A and then that $H \simeq Z_q \wr A$ with Z_q corresponding to the cyclic group $\langle y_0\rangle$. It follows that $\mathfrak{Z}(H)$, the center of H, is the cyclic group of order q generated by $z = y_0^{x_0}y_0^{x_1}\cdots y_0^{x_n}$.

Let i be fixed and let $B_i = \{(b, x_i) | b \in B\}$. Since B is abelian the map $B \to B_i$ given by $b \to (b, x_i)$ is easily seen to be a homomorphism onto. Hence B_i is a subgroup of B and $B_i \simeq B/\mathbf{C}_B(x_i)$ since $\mathbf{C}_B(x_i)$ is clearly the kernel of the homomorphism. It then follows easily in $K[G]$ that

$$\sum_{b\in B} (x_i^{-1})^b = \sum_{b\in B} (b, x_i)x_i^{-1} = |\mathbf{C}_B(x_i)|\hat{B}_i x_i^{-1}$$

where $\hat{B}_i$ denotes the sum of the elements of B_i.

Now let $\alpha = \sum k_i x_i^{-1} \in I \cap K[A]$ with $k_0 \neq 0$. Then $\beta = \sum_{b\in B} b^{-1}\alpha b$

$\in I$ and by the above

$$\beta = \sum_i k_i |\mathbf{C}_B(x_i)| \hat{B}_i x_i^{-1}.$$

For $i = 0$, since $x_0 = 1$ the summand here is $k_0|B|x_0^{-1} = k_0|B|$. On the other hand if $i \neq 0$ then x_i does not centralize B so B_i is a nonidentity subgroup of B. Furthermore since both A and B are abelian, we conclude easily that $B_i \triangle H$ and hence since H is a q-group this yields $B_i \cap \mathfrak{Z}(H) \neq \langle 1 \rangle$. Thus $z \in B_i$ so $(z - 1)\hat{B}_i = 0$ and we have

$$(z - 1)\beta = k_0|B|(z - 1).$$

Finally $(z - 1)\beta \in I$, $k_0 \neq 0$ and $|B| \neq 0$ in K since by assumption $q \neq \operatorname{char} K$. We have therefore shown that there exists a nonidentity element $z \in G$ with $z - 1 \in I$.

Step 3. Let $H = \{h \in G | h - 1 \in I\}$. Then since I is an ideal it follows that H is a normal subgroup of G. Furthermore by Step 2 we have $H \neq \langle 1 \rangle$. Hence since G is simple we conclude that $H = G$ and this implies immediately that $I \supseteq \omega(K[G])$. But $\omega(K[G])$ is a maximal ideal of $K[G]$ so this yields $I = \omega(K[G])$ and we have therefore obtained our main assertion.

Finally apply the hypothesis of the theorem with $1 = x_0$. Then by (3), G has a cyclic subgroup C of order q and hence $e = 1 - \hat{C}/q$ is a nonzero idempotent of $K[G]$. Furthermore, nonzero idempotents are never contained in the Jacobson radical of a ring so there exists an irreducible $K[G]$-module V with $Ve \neq 0$. Since $e \in \omega(K[G])$ this yields $V\omega(K[G]) \neq 0$ and thus the zero ideal is the only possibility for the kernel of the action of $K[G]$ on V. This means that V is a faithful irreducible $K[G]$-module so $K[G]$ is primitive and the Theorem is proved. □

To see where the above elements y_i might come from, we make the following simple observation.

LEMMA. *Let $1 = x_0, x_1, \ldots, x_n$ be distinct elements of G and let $A = \langle y_0, y_1, \ldots, y_n \rangle$ be an elementary abelian q-group of order q^{n+1}. If A and G are suitably embedded in the wreath product $A \wr G$, then the elements x_i and y_i satisfy conditions* (1), (2) *and* (3) *of the Theorem.*

We now use this to handle some interesting families of groups. A group G is said to be algebraically closed [3] if every finite system $W_i(x_j, y_k) = 1$ and $\overline{W}_i(x_j, y_k) \neq 1$ of word equations and word inequalities, in the variables y_k and group elements x_j, which has a simultaneous solution in some group extension of G also has a solution in G. Such groups are quite plentiful and in fact, by [3, Theorem 1], every infinite group can be embedded in an algebraically closed group of the same cardinality.

COROLLARY 1. *Let G be an algebraically closed group and let K be a field. Then $\omega(K[G])$ is the unique proper ideal of $K[G]$ and the group ring is primitive.*

PROOF. By [2], G is a simple group. Fix a prime q different from the characteristic of K and let $1 = x_0, x_1, \ldots, x_n$ be finitely many distinct elements of G. Then by the Lemma there exists a group extension of G having elements $y_0, y_1, \ldots, y_n$ satisfying conditions (1), (2) and (3) of the Theorem.

But observe that condition (1) is merely a finite set of commuting and order equations and that, given (1), conditions (2) and (3) amount to a finite set of inequalities. Thus since G is algebraically closed these equations and inequalities must also have a solution in G. Thus the Theorem applies and the result follows. □

Other groups of interest are the universal groups of Ph. Hall. A group G is universal (see [1, Chapter 6]) if it is locally finite, contains copies of all finite groups and has the property that any two isomorphic finite subgroups are conjugate. Such groups are reasonably numerous and indeed, by [1, Theorem 6.5], every infinite locally finite group can be embedded in a universal group of the same cardinality.

COROLLARY 2. *Let G be a universal group and let K be a field. Then $\omega(K[G])$ is the unique proper ideal of $K[G]$ and the group ring is primitive.*

PROOF. By [1, Theorem 6.1(d)] G is simple. Fix a prime q different from the characteristic of K and let $1 = x_0, x_1, \ldots, x_n$ be finitely many distinct elements of G. Then $H = \langle x_0, x_1, \ldots, x_n \rangle$ is a finite group, since G is locally finite. By the Lemma, if $A = \langle y_0, y_1, \ldots, y_n \rangle$ is an elementary abelian q-group of order q^{n+1} then the elements x_i and y_i in $A \wr H$ satisfy properties (1), (2) and (3) of the Theorem. But by [1, Theorem 6.1(b)] the embedding of H into G can be extended to an embedding of $A \wr H$ into G. Therefore G satisfies the hypothesis of the Theorem and the result follows. □

Finally let G be an arbitrary group. If H is a proper normal subgroup of G, then $I = \omega(K[H]) \cdot K[G]$ is a proper ideal of $K[G]$ distinct from the augmentation ideal. Thus a necessary condition for $\omega(K[G])$ to be the unique proper ideal of the group ring is that G be simple. On the other hand this condition is by no means sufficient. Consider for example $G = \mathrm{Alt}_\Omega$ where Ω is an infinite set and each element of G moves only finitely many points. Of course G is simple. Let K by any field and form the permutation module V for $K[G]$. That is, V has as a K-basis the elements of Ω and G acts on V by appropriately permuting this basis. If σ and τ are two disjoint permutations in G, for example take $\sigma = (123)$ and $\tau = (456)$, then it is easy to see that $(\sigma - 1)(\tau - 1) \neq 0$ acts trivially on V but that $\sigma - 1$ does not. Hence the kernel of the action of $K[G]$ on V is a proper ideal different from $\omega(K[G])$.

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