

TEST MODULES AND COGENERATORS

PETER VÁMOS

ABSTRACT. If $\text{Hom}_R(A, T) = 0$ implies that $A = 0$ for all R -modules A , then the R -module T is a *test module*. The ring R is said to be a *TC-ring* if every test module is a cogenerator. If S is a simple module over a *TC-ring* then $\text{End}_R E(S)$ is a local semifir. A commutative ring R is a *TC-ring* if and only if R_M is a P.I.D. for all maximal ideals M of R .

All rings have an identity element and modules are unitary left modules. For an R -module A , $E(A)$ denotes the injective envelope of A and A^k the k -fold direct sum of A . Let R be a ring and T an R -module. We say that T is a *test module* if $\text{Hom}_R(A, T) = 0$ implies $A = 0$ for all R -modules A . The following result was proved by T. Cheatham and R. Cumbie.

THEOREM A [1, THEOREM 3]. *For a ring R the following are equivalent:*

- (i) *every test module is a cogenerator;*
- (ii) *for all simple modules S , $E(S)$ is contained in every nonzero factor of $E(S)$.*

A ring satisfying conditions (i) and (ii) in Theorem A is called a *TC-ring*.

Let R be a *TC-ring*, S a simple R -module and set $E = E(S)$. Some of the properties of E which follow easily from Theorem A are set out in the following two lemmas.

LEMMA 1. *Let A be a nonzero factor module of E . Then the following assertions are true:*

- (i) *the socle of A is a direct sum of copies of S and A is an essential extension of its socle;*
- (ii) *if the socle of A is isomorphic to S^k for some integer $k > 0$, then $A \approx E^k$;*
- (iii) *if $f \in \text{End}_R E$ and $f \neq 0$ then f is onto;*
- (iv) *if S' is another simple R -module such that $S \approx S'$, then $\text{Hom}_R(E, E(S')) = 0$.*

PROOF. Property (i) was noticed in [1] and is a simple consequence of condition (ii) in Theorem A. Now assume that the socle of A is S^k for some $k > 0$. Since A contains E we have $A = E \oplus A'$ and $A' \neq 0$ if $k > 1$. Since A' is again a factor of E it has, in turn, a summand isomorphic to E if $k > 1$. Continuing this we obtain $A \approx E^k$. Next, let $f \in \text{End}_R E, f \neq 0$. Then $f(E) \subseteq E$ and $f(E)$ contains a copy of E . Since E is indecomposable, $f(E) = E$. Finally (iv) is immediate from (i).

Received by the editors November 10, 1974 and, in revised form, September 11, 1975.

AMS (MOS) subject classifications (1970). Primary 13C10; Secondary 16A06.

Key words and phrases. Test module, cogenerator, indecomposable injective, endomorphism ring, semifir, P.I.D.

© American Mathematical Society 1976

We remark that in view of (iv) in Lemma 1 the ‘ cofinitely ’ generated modules over a TC-ring split into their ‘ homogeneous ’ components. For the details the reader is referred to [2, Theorem 4.26].

LEMMA 2. Let $Q = \text{End}_R E, f_1, \dots, f_k \in Q$ and assume that the intersection $\text{Ker } f_1 \cap \dots \cap \text{Ker } f_k$ is irredundant. Then the natural monomorphism

$$h: E/(\text{Ker } f_1 \cap \dots \cap \text{Ker } f_k) \rightarrow E/\text{Ker } f_1 \oplus \dots \oplus E/\text{Ker } f_k$$

is an isomorphism. In particular $\text{Ker } f_i + (\bigcap_{j \neq i} \text{Ker } f_j) = E$ for all $1 \leq i \leq k$.

PROOF. It is well known that h is an essential monomorphism (e.g. [2, Theorem 4.9]). Also, $E/(\text{Ker } f_1 \cap \dots \cap \text{Ker } f_k)$ is injective by (ii) of Lemma 1. Thus h is an isomorphism. The second part of Lemma 2 now follows from the fact that $h(\text{Ker } f_i) = \bigoplus_{j \neq i} E/\text{Ker } f_j$ and $h(\bigcap_{j \neq i} \text{Ker } f_j) = E/\text{Ker } f_i$.

By a local ring we mean a ring with a unique maximal (left) ideal.

PROPOSITION 3. The ring of endomorphisms Q of E is a local semifir i.e. Q is a local domain and every finitely generated (left) ideal of Q is free.

PROOF. Since E is indecomposable, Q is local and (iii) of Lemma 1 implies that Q is a domain. Let $L = Qf_1 + \dots + Qf_n, f_1, \dots, f_n \in Q$ be a left ideal of Q . We may assume that $L \neq 0$ and that $\text{Ker } f_1 \cap \dots \cap \text{Ker } f_n = \text{Ker } f_1 \cap \dots \cap \text{Ker } f_k$ for some $1 \leq k \leq n$ with $\text{Ker } f_1 \cap \dots \cap \text{Ker } f_k$ irredundant. If $f \in L$ then $\text{Ker } f \supseteq \text{Ker } f_1 \cap \dots \cap \text{Ker } f_k$ and a standard argument shows the existence of $g_1, \dots, g_k \in Q$ such that $f = g_1f_1 + \dots + g_kf_k$. (See e.g. [2, Theorem 5.15, p. 144].) Hence $L = Qf_1 + \dots + Qf_k$. We claim L is free on f_1, \dots, f_k . Let $g_1f_1 + \dots + g_kf_k = 0, g_1, \dots, g_k \in Q$ and suppose that $g_i(a) \neq 0$ for some $a \in E$ and $1 \leq i \leq k$. Put $A = \bigcap_{j \neq i} \text{Ker } f_j$. Then $E = \text{Ker } f_i + A$ by Lemma 2 and $f_i(A) = E$. Now $a = f_i(b)$ for some $b \in A$ and $0 = (g_1f_1 + \dots + g_kf_k)(b) = g_if_i(b) = g_i(a) \neq 0$. Thus $g_i = 0$ for all $1 \leq i \leq k$ and L is free.

THEOREM 4. For a commutative ring R the following are equivalent:

- (i) R is a TC-ring;
- (ii) R_M is a P.I.D. for all maximal ideals M of R ;
- (iii) every nonzero factor module of $E(R/M)$ is isomorphic to $E(R/M)$ for all maximal ideals M of R .

PROOF. The implication (iii) \Rightarrow (i) is clear and the equivalence of conditions (ii) and (iii) is a special case of the Corollary in [3].

Assume (i) and let M be a maximal ideal of R . Set $E = E(R/M)$. Then E is, in a natural way, an R_M module and it is the R_M -injective envelope of the only simple R_M module $R/M \approx R_M/R_M M$. Then R_M is also a TC-ring by (ii) of Theorem A. In other words we may assume that R is a local ring with maximal ideal M . Then E is a cogenerator and we may assume that R is a subring of $\text{End}_R E$. Hence R is a domain by Proposition 3. Let $f, g \in R$. We want to show that either $Rf \supseteq Rg$ or $Rg \supseteq Rf$. We may assume that $fg \neq 0$. If $\text{Ker } f \cap \text{Ker } g$ were irredundant then $E = \text{Ker } f + \text{Ker } g$ by Lemma 2. But $\text{Ker } fg \supseteq \text{Ker } f + \text{Ker } g$ and $fg \neq 0$. Therefore either $\text{Ker } f \supseteq \text{Ker } g$ or $\text{Ker } g \supseteq \text{Ker } f$. Next, the fact that E is the minimal injective cogenerator implies that $\text{Ann}_R \text{Ann}_E L = L$ for all

ideals of L of R (see [2, p. 148, (5.4.3)]). This, in turn, yields $Rf \supseteq Rg$ or $Rg \supseteq Rf$. Thus R is a valuation domain. Let L be a nonzero ideal of R and $a \in L$, $a \neq 0$. Set $K = Ma$. Since the ideals of R are totally ordered we have $K \subseteq L$, K is irreducible and R/K is an essential extension of $Ra/K \approx R/M$. Therefore $\bar{E}(R/K) \approx E$ and the natural epimorphism $R/K \rightarrow R/L$ extends to a homomorphism $E \rightarrow E(R/L)$. This shows that R/L is contained in a nonzero factor of E and the socle of R/L is nonzero by Lemma 1. Since L is irreducible, we obtain that $E(R/L) \approx E$. Accordingly, R/L is Artinian for all nonzero ideals of R by [2, Theorem 3.21]. But this can only happen if R is a field or a rank-one discrete valuation ring. In either case, R is a P.I.D.

REFERENCES

1. T. Cheatham and R. Cumbie, *Test modules*, Proc. Amer. Math. Soc. **49** (1975), 311–314.
2. D. W. Sharpe and P. Vámos, *Injective modules*, Cambridge Univ. Press, New York, 1972.
3. P. Vámos, *A note on the quotients of indecomposable injective modules*, Canad. Math. Bull. **12** (1969), 661–665. MR **41** #190.

DEPARTMENT OF PURE MATHEMATICS, THE UNIVERSITY OF SHEFFIELD, SHEFFIELD 10, ENGLAND