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STRONGLY HOMOGENEOUS TORSION FREE ABELIAN GROUPS OF FINITE RANK

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ABSTRACT. An abelian group is *strongly homogeneous* if for any two pure rank 1 subgroups there is an automorphism sending one onto the other. Finite rank torsion free strongly homogeneous groups are characterized as the tensor product of certain subrings of algebraic number fields with finite direct sums of isomorphic subgroups of Q, the additive group of rationals. If G is a finite direct sum of finite rank torsion free strongly homogeneous groups, then any two decompositions of G into a direct sum of indecomposable subgroups are equivalent.

D. K. Harrison, in an unpublished note, defined a *p-special* group to be a strongly homogeneous group such that $G/pG \simeq Z/pZ$ for some prime p and $qG = G$ for all primes $q \neq p$ and characterized these groups as the additive groups of certain valuation rings in algebraic number fields. Richman [7] provided a global version of this result. Call G *special* if G is strongly homogeneous, $G/pG = 0$ or Z/pZ for all primes p, and G contains a pure rank 1 subgroup isomorphic to a subring of Q. Special groups are then characterized as additive subgroups of the intersection of certain valuation rings in an algebraic number field (also see Murley [5]). Strongly homogeneous groups of rank 2 are characterized in [2]. All of the above-mentioned characterizations can be derived from the more general (notation and terminology are as in Fuchs [3]):

THEOREM 1. *Let G be a torsion free abelian group of finite rank. Then G is strongly homogeneous iff $G \simeq R \otimes_Z H$ where H is a finite direct sum of isomorphic torsion free abelian groups of rank* 1, *R is a subring of an algebraic number field K (with $1_K \in R$), and every element of R is an integral multiple of a unit in R.*

PROOF. ($\Leftarrow$) First of all, the additive group of R, denoted by R^+, is strongly homogeneous. Let X and Y be pure rank 1 subgroups of R^+. There are units u and v of R in X and Y, respectively. Left multiplication by vu^{-1} induces an automorphism g of R^+ with $g(X) = Y$.

Secondly, $R \otimes_Z A$ is strongly homogeneous, where A is a torsion free abelian group of rank 1 with $H \simeq \sum_{i=1}^{k} \oplus A$. Choose $0 \neq a \in A$ and define $\delta: R^+ \to R \otimes_Z A$ by $\delta(r) = r \otimes a$. Then δ is a monomorphism. Let X and Y

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be pure rank 1 subgroups of $R \otimes_Z A$, and define $X' = (\text{image } \delta) \cap X$ and $Y' = (\text{image } \delta) \cap Y$. By the preceding remarks there is a unit $u \in R$ such that left multiplication by u induces an automorphism g of image δ and $g(X') = Y'$. Therefore, left multiplication by u induces an automorphism h of $R \otimes_Z A$ extending g. Since $\text{rank}(A) = 1$, $(R \otimes_Z A)/\text{image } \delta$ is torsion. Consequently, $h(X) \subseteq Y$, for if $x \in X$, then $mx \in X'$ for some $0 \neq m \in Z$; $mh(x) = h(mx) = g(mx) \in Y' \subseteq Y$; and Y is pure in $R \otimes_Z A$. Thus $h(X) = Y$ since h is an automorphism.

Finally, $G = R \otimes_Z H$ is strongly homogeneous. Note that R is a principal ideal domain since every ideal of R is of the form nR for some $n \in Z$ (since R is a subring of an algebraic number field; every ideal is finitely generated; and every element of R is an integral multiple of a unit in R). Also, $R \otimes_Z A$ is an R-module of R-rank 1 so that G may be regarded as an R-direct sum of isomorphic torsion free R-modules of R-rank 1. Consequently, every R-pure submodule of G is a direct summand of G (see Fuchs [3, p. 115] for the case that $R = Z$ and observe that the same proof is true in this case).

Suppose that X is a pure rank 1 subgroup of G and let B be the pure subgroup of G generated by RX. Then B is an R-submodule of G, for if $y \in B$, then $my \in RX$ for some $0 \neq m \in Z$ and $mRy \subseteq RX \subseteq B$. In fact, B is an R-pure submodule of G, since if $r = nu \in R$ with $n \in Z$ and u a unit of R, then $rG \cap B = nG \cap B = nB = rB$. Consequently, B is an R-summand of G with R-rank $B = 1$. As a consequence of the argument given in Fuchs [3, p. 114], $B \simeq R \otimes_Z A$ since $G \simeq \Sigma_{i=1}^{k} \oplus (R \otimes_Z C)$. Write $\langle X \rangle_*$ for the pure subgroup of G generated by X.

Now let X and Y be two pure rank 1 subgroups of G. Then $\langle RX \rangle_* \simeq (R \otimes_Z A) \simeq \langle RY \rangle_*$: $\langle RX \rangle_*$ and $\langle RY \rangle_*$ are summands of G; and $R \otimes_Z A$ is strongly homogeneous so there is an automorphism g of G with $g(X) = Y$.

($\Rightarrow$) We first prove that G is *irreducible* (i.e., if P is a pure fully invariant subgroup of G, then $P = 0$ or G). Let $0 \neq P$ be a pure fully invariant subgroup of G; X a pure rank 1 subgroup of P; and Y a pure rank 1 subgroup of G. There is an automorphism g of G with $Y = g(X) \subseteq g(P) \subseteq P$. Therefore, $P = G$. As a consequence of Reid [6], there is $0 \neq m \in Z$ with $mG \subseteq G_1 \oplus G_2 \oplus \cdots \oplus G_k \subseteq G$, where $Q \otimes_Z \text{End}(G_i)$ is a division algebra with Q-dimension equal to rank G_i and there are monomorphisms f_{ij}: $G_i \to G_j$ with $G_j/\text{image} f_{ij}$ bounded for all $1 \leqslant i, j \leqslant k$. Since $m\langle G_i \rangle_* \subseteq G_i \subseteq \langle G_i \rangle_*$, for each i, it is sufficient to assume that each G_i is pure in G.

Let X_i and X_j be pure rank 1 subgroups of G_i and G_j, respectively. There is an automorphism g of G with $g(X_i) = X_j$. For each $1 \leqslant l \leqslant k$, let Π_l: $G \to G_l$ be a *quasi-projection* (i.e., multiplication by m followed by a projection onto G_l). Define g_l to be $\Pi_l g$, restricted to G_i. If $l \neq j$ then $g_l(X_i) = 0$; $f_{li} g_l \in \text{End}(G_i)$; $f_{li} g_l = 0$ since $Q \otimes_Z \text{End}(G_i)$ is a division algebra and $f_{li} g_l(X_i) = 0$; and $g_l = 0$ since f_{li} is a monomorphism. But

$$\begin{aligned} mg(G_i) &= \Pi_1 g(G_i) + \cdots + \Pi_k g(G_i) \\ &= g_1(G_i) + \cdots + g_k(G_j) = g_j(G_i) \subseteq G_j \end{aligned}$$

so that $g(G_i) \subseteq G_j$ since G_j is pure in G. Since g is an automorphism of G, it follows that $g(G_i) = G_j$.

In particular, $G_i \simeq G_j$ for all $1 \leqslant i, j \leqslant k$ and each G_i is strongly homogeneous. Let $A = G_1$, $R = \text{End}(A)$, and X a pure rank 1 subgroup of A. Define $\delta\colon R \otimes_Z X \to A$ by $\delta(r \otimes x) = rx$. Since A is strongly homogeneous, δ is an epimorphism. But

$$\text{rank}(A) = Q\text{-dimension } Q \otimes_Z \text{End}(G_1) = \text{rank}(R) = \text{rank}(R \otimes_Z X)$$

so that δ is an isomorphism.

Next, we prove that if $0 \neq r \in R$, then r is an integral multiple of a unit in R. Let X be a pure rank 1 subgroup of A such that $Y = \langle r(X)\rangle_*$ is a pure rank 1 subgroup of A. Since A is strongly homogeneous, there is a unit $u \in R$ with $u(X) = Y$. The restriction of r and u to X are nonzero elements of the rank 1 group $\text{Hom}(X, Y)$ so there exist relatively prime integers c and d with $cu(x) = dr(x)$ for all $x \in X$. Then $dr(X) = cu(X) = cY$, and it follows that $dY = Y$. Since any two pure rank 1 subgroups of A are isomorphic, $dA = A$; u/d is an automorphism of A and $(c(u/d) - r)(X) = 0$. But $Q \otimes_Z R$ is a division algebra so $c(u/d) = r$ with u/d a unit of R and $c \in Z$ as desired.

Since $R = \text{End}(A)$, $R \otimes_Z X \simeq A$ for a rank 1 group X, and $Q \otimes_Z R$ is a division algebra, it follows that $R \simeq \text{End}(R^+)$. Embed R^+ in $Q \otimes_Z R$ by $r \to 1 \otimes r$ so that $R = \{\alpha \in Q \otimes_Z R \mid \alpha R^+ \subseteq R^+\}$. It follows that R is commutative and that R is a principal ideal domain.

Finally, $mG \subseteq G_1 \oplus \cdots \oplus G_k \subseteq G$; each G_i is isomorphic to A; $\text{End}(A)$ is a principal ideal domain; and since G is strongly homogeneous the map $\text{Hom}(A,G) \otimes_Z A \to G$ is an epimorphism. Therefore, G is isomorphic to a finite direct sum of copies of A (Arnold and Lady [1]). Since $A \simeq R \otimes_Z X$, for a pure rank 1 subgroup X of A, $G \simeq R \otimes_Z H$ where $H = \sum_{i=1}^{k} \oplus X$, completing the proof.

The following corollary summarizes some consequences of the proof of Theorem 1.

COROLLARY 2. *Let G be a strongly homogeneous torsion free abelian group of finite rank.*

(a) $G \simeq \sum_{i=1}^{k} \oplus A$ *where A is strongly homogeneous and strongly indecomposable (i.e. $mA \subseteq A_1 \oplus A_2 \subseteq A$ for some $0 \neq m \in Z$ implies that $A_1 = 0$ or $A_2 = 0$).*

(b) *G is indecomposable iff G is strongly indecomposable.*

(c) *If B is a direct summand of G then $B \simeq \sum_{i=1}^{l} \oplus A$.*

(d) *If H is another strongly homogeneous torsion free abelian group of finite rank, then $G \simeq H$ iff* rank G = rank H, *G and H have isomorphic pure rank* 1 *subgroups and the center of* $\text{End}(G)$ *is isomorphic to the center of* $\text{End}(H)$.

PROOF. A consequence of the following observations:

(i) $G \simeq \sum_{i=1}^{k} \oplus A$, where A is strongly homogeneous and $Q \otimes_Z \text{End}(A)$ is a field (thus A is strongly indecomposable);

(ii) $\text{End}(A)$ is a principal ideal domain so that any summand of G is isomorphic to a direct sum of copies of A (Arnold and Lady [1]); $A \simeq \text{End}(A) \otimes_Z X$, where X is a pure rank 1 subgroup of A; and the center of $\text{End}(G)$ is isomorphic to $\text{End}(A)$.

Two finite rank torsion free abelian groups G and H are *quasi-isomorphic* if there is a monomorphism $f\colon G \to H$ such that $H/\text{image } f$ is bounded.

PROPOSITION 3. *Suppose that G and H are two strongly homogeneous torsion free abelian groups of finite rank. Then G and H are quasi-isomorphic iff G and H are isomorphic.*

PROOF. Write $G = \sum_{i=1}^{k} \oplus A$ and $H = \sum_{i=1}^{l} \oplus C$ where A and C are strongly homogeneous and strongly indecomposable. If G and H are quasi-isomorphic, then rank G = rank H and the groups G and H have isomorphic pure rank 1 subgroups. In view of Corollary 2(d), it is sufficient to prove that $\mathrm{End}(A) \simeq \mathrm{End}(C)$. Since A and C are strongly indecomposable, the Krull-Schmidt theorem for quasi-decompositions (e.g., see Reid [6]) guarantees that A and C are quasi-isomorphic.

Let m be the least positive integer such that there is a monomorphism ϕ: $A \to C$ with $mC \subseteq \phi(A) = B \subseteq C$. Recall that every endomorphism of A (or C) is an integral multiple of an automorphism.

Suppose that f is an automorphism of B, embed C in $Q \otimes_Z C$ so that $QB = Q \otimes_Z C$, extend f to $Q \otimes_Z C$, and let n be the least positive integer with $nf(C) \subseteq C$. Such an n exists since $mf(C) = f(mC) \in f(B) = B \subseteq C$. Then $nf = ku$ for some $k \in Z$, u an automorphism of C, and $pC \neq C$ for all primes p dividing k. If d = g.c.d.(n,k) then $(n/d)f(C) = (k/d)u(C) \subseteq C$ so that, by the minimality of $n,d = 1$. Thus $nB = nf(B) = ku(B) \subseteq kC$. Since g.c.d.$(n,k) = 1$, $mC \subseteq B \subseteq kC$. By the minimality of m, $k = 1$, since otherwise there is a monomorphism ϕ: $A \to C$ with $lC \subseteq \phi(A) \subseteq C$ and $l < m$. Therefore, $nf = u$, $u(B) = nf(B) = nB$, and $B = u^{-1}u(B) = nu^{-1}(B) \subseteq nC$. Again by the minimality of m, $n = 1$.

Every endomorphism of B is an integral multiple of an automorphism of B so, by the preceding remarks, there is a homomorphism θ: $\mathrm{End}(B) \to \mathrm{End}(C)$ given by lifting endomorphisms of B to $Q \otimes_Z C$ and restricting to C.

It is now sufficient to prove that θ is onto. Let u be an automorphism of C and n the least positive integer with $nu(B) \subseteq B$. There is such an n since $mu(B) \subseteq mC \subseteq B$. Write $nu = kf$ for $k \in Z$ and f an automorphism of B. By the minimality of n, g.c.d.$(k,n) = 1$. Therefore, $kB = kf(B) = nu(B) \subseteq nC$. It follows that $mC \subseteq B \subseteq nC$ and that $n = 1$ by the minimality of m. Thus $u = kf$ and θ is onto, as desired.

COROLLARY 4. *Let $\mathfrak{S}$ be the class of finite direct sums of strongly homogeneous torsion free abelian groups of finite rank.*

(a) *If G is in $\mathfrak{S}$ and B is a summand of G then B is in $\mathfrak{S}$.*

(b) *Every G in $\mathfrak{S}$ has the Krull-Schmidt property* (*i.e. any two decompositions of G into direct sums of indecomposable groups are equivalent*).

PROOF. A consequence of Corollary 2, Proposition 3, and a result of Arnold and Lady [1]. Briefly, assume that B is indecomposable, write $G = \sum \oplus A_i$ where each A_i is strongly indecomposable and strongly homogeneous, let $A = \sum \oplus \{A_i | A_i \simeq A_1\}$ and $K = \sum \oplus \{A_i | A_i \not\simeq A_1\}$ so that $G = A \oplus K = B \oplus C$. Now no quasi-summand of A is quasi-isomorphic to any quasi-summand of K since A_i and A_j are quasi-isomorphic iff $A_i \simeq A_j$. By Arnold and Lady [1], $B = B_1 \oplus B_2$, $C = C_1 \oplus C_2$ and $G = B_1 \oplus C_1 \oplus K = A \oplus B_2 \oplus C_2$. Therefore, either $B = B_1 \simeq A_1$ or $B = B_2$, $A \simeq C_1$, $H = A \oplus K = C_1 \oplus (C_2 \oplus B)$ and $K \simeq C_2 \oplus B$. By induction, B is strongly homogeneous.

For the following corollaries we assume fundamental properties of subrings of algebraic number fields as, for example, in Kaplansky [4] or Samuel [8].

PROPOSITION 5. *Let R be an integral domain such that the quotient field K of R is an algebraic number field. Then every element of R is an integral multiple of a unit in R iff $R = \bigcap \{J_P | P \in S\}$ where J is the ring of algebraic integers of K, S is a set of prime ideals of J, and if p is a rational prime with $pJ = P_1^{e_1}P_2^{e_2} \cdots P_n^{e_n}$, a product of powers of distinct prime ideals of J, then at most one $P_i \in S$ and $P_i \in S$ implies that $e_i = 1$.*

PROOF. ($\Leftarrow$) If p is a rational prime with $pR \neq R$, then $pJ_P \neq J_P$ for some $P \in S$. Furthermore, $pJ_P = PJ_P$ and $pJ_{P'} = J_{P'}$ for all $P' \in S \setminus \{P\}$. Therefore, $pR = pJ_P \cap R = PJ_P \cap R$ and the canonical map $R/pR \to J_P/pJ_P$ is a monomorphism. Thus, R/pR is a field; every prime ideal R is of the form pR for a rational prime p; and every ideal I of R can be written as nR for some $n \in Z$ (since R is Dedekind, every ideal is a product of prime ideals). Consequently, every element of R is an integral multiple of a unit of R.

($\Rightarrow$) Since R is a principal ideal domain, R is integrally closed in K. Consequently, $J \subseteq R$. But $\{pR | p$ is a rational prime and $pR \neq R\}$ is precisely the set of prime ideals of R. Each $R_{pR} = J_P$ for some prime ideal P of J so that $PJ_P = pJ_P$ and $R/pR = J_P/pJ_P$ is a field. Therefore, if $pJ = P_1^{e_1}P_2^{e_2} \cdots P_n^{e_n}$, then $P = P_i$ for exactly one i and $e_i = 1$ since $J_P/pJ_P \simeq J_{P_i}/P_i^{e_i}J_{P_i}$ is a field.

COROLLARY 6. *Suppose that G is an indecomposable torsion free abelian group of finite rank. Then G is strongly homogeneous with a pure rank 1 subgroup isomorphic to a subring of Q iff G is isomorphic to the additive group of a ring R satisfying the condition of Proposition 5.*

PROOF. Since G is indecomposable, $G \simeq R \otimes_Z H$ for some rank 1 group H and some R such that every element of R is an integral multiple of a unit of R (Theorem 1).

Since G is strongly homogeneous and H is isomorphic to a pure rank 1 subgroup of G, H must be isomorphic to a subring of Q. As in the proof of Theorem 1, $R \simeq \text{End}(G)$ so that H is isomorphic to a subring of R (namely $Q \cap R$). Therefore $R \otimes_Z H$ is isomorphic, as a ring, to R and the proof is complete.

COROLLARY 7. *Let G be a torsion free abelian group of finite rank.*

(a) (Harrison) *G is p-special iff $G \simeq J_P$, where J is the ring of algebraic integers of an algebraic number field, $pJ = P_1P_2^{e_2} \cdots P_n^{e_n}$ a product of powers of distinct prime ideals of J, $P = P_1$, and $J/P \simeq Z/pZ$.*

(b) (Richman [7]) *G is special iff $G = \bigcap_{P \in S} J_P$ where J is the ring of algebraic integers of an algebraic number field; if $pJ = P_1^{e_1}P_2^{e_2} \cdots P_n^{e_n}$, then at most one $P_i \in S$, and if $P_i \in S$ then $e_i = 1$ and $J/P_i \simeq Z/pZ$.*

(c) (Arnold, Vinsonhaler and Wickless [2]) *If* rank $G = 2$ *then G is strongly homogeneous iff either $G \simeq A \oplus A$ or* (i) $Q \otimes_Z \text{End}(G) = Q(\sqrt{n})$ *for some square free integer N;* (ii) *Z/pZ – dimension of $G/pG \leqslant 1$ for all odd primes p such that N is a quadratic residue* mod p; *$Z/2Z$ – dimension of $G/2G \leqslant 1$ if $N \equiv 1$* (mod 8); *$2G = G$ if $N \equiv 2$ or* 3 (mod 4); *and $NG = G$.*

PROOF. (a), (b) Apply Corollaries 5 and 6 noting that if G is strongly homogeneous and Z/pZ − dimension $G/pG = 1$ for some rational prime p then G must be indecomposable.

(c) Follows from Theorem 1, Corollary 5, Corollary 6 and the following: let p be a rational prime; then p odd and N a quadratic residue mod p imply that p splits in $Q(\sqrt{N})$; p an odd divisor of N implies that p ramifies in $Q(\sqrt{N})$; if $N \equiv 1 \pmod 8$, then 2 splits in $Q(\sqrt{N})$; and if $N \equiv 2$ or $3 \pmod 4$, then 2 ramifies in $Q(\sqrt{N})$ (e.g., see Samuel [8]).

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