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$L^2(G_{\mathbf{Q}}\backslash G_{\mathbf{A}})$ IS NOT ALWAYS MULTIPLICITY-FREE

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ABSTRACT. We show that there are solvable adelic groups $G_{\mathbf{A}}$ whose action on $L^2(G_{\mathbf{Q}}\backslash G_{\mathbf{A}})$, $G_{\mathbf{Q}}\backslash G_{\mathbf{A}}$ compact, it is not multiplicity-free.

In the course of this work, we shall have to review certain methods for computing these multiplicities. These methods are described in, e.g., [2] and [4], but are found (at least implicitly) in various other papers as well.

In the counterexample, we let G be the semidirect product of a (normal) Abelian group N by an Abelian group K; let $\Gamma_0 = N \cap \Gamma$, $\Delta = K \cap \Gamma$. Γ_0 will, in fact, be a finite-dimensional vector space over $\mathbf{Q}$, and N will be $\Gamma \otimes_{\mathbf{Q}} A$. Poisson summation says that the action of N on $\mathfrak{L}^2(\Gamma\backslash N)$ decomposes into $\oplus_{\chi \in \Gamma_0^\perp} \chi$, a simple sum of characters. K acts on $\hat{N}$ by $\chi^k(x) = \chi(k^{-1}xk)$; let H be the subgroup of K fixing χ, and let $\Delta_0 = H \cap \Delta$.

LEMMA 1. *$\Delta_0\backslash H$ is compact.*

PROOF. The orbit of χ_1 under Δ is a continuous 1-1 image of $H\backslash H\Delta$. On the other hand, every element of Δ acts as an automorphism of Γ_0 and thus takes $\Gamma_0^\perp$ to $\Gamma_0^\perp$. Hence, $\chi^\Delta \subseteq \Gamma_0^\perp$ is discrete, and therefore $H\backslash H\Delta$ is discrete—in particular, closed in $H\backslash K$. If $\Delta_0\backslash H$ is not compact, then, since $\Delta_0\backslash H \cong \Delta\backslash H\Delta$, $\Delta\backslash H\Delta$ is not compact in $\Delta\backslash K$; hence, $H\Delta$ is not closed in K, and so $H\backslash H\Delta$ is not closed in $H\backslash K$. The lemma follows.

According to [5], the irreducible representations of NH lying over χ_1 are of the form $\chi_1 \otimes \psi$, where ψ is an element of $\hat{H}$. From this, it is easy to see that the irreducible representations of NH lying over χ_1 in $\mathfrak{L}^2(\Gamma\Delta_0\backslash NH)$ are of the form $\chi_1 \otimes \psi_1$, where $\psi_1 \in \Delta_0^\perp \subseteq \hat{H}$. Note also that $\mathfrak{L}^2(\Gamma\Delta_0\backslash NH) \cong \mathfrak{L}^2(\Gamma\Delta\backslash NH\Delta)$, since $\Gamma\Delta_0\backslash NH \cong \Gamma\Delta\backslash NH\Delta$; thus Δ acts on $\mathfrak{L}^2(\Gamma\Delta_0\backslash NH)$. Furthermore, if $\delta \in \Delta$, then the subgroup of K fixing χ_1^δ is $\delta^{-1}H\delta = H$, and the representations lying above χ_1^δ in $\mathfrak{L}^2(\Gamma\Delta_0\backslash NH)$ are those of the form $\chi_1^\delta \otimes \psi_1$, $\psi_1 \in \Delta_0^\perp$. The action of δ on $\mathfrak{L}^2(\Gamma\Delta_0\backslash NH)$ takes $\chi_1 \otimes \psi_1$ to the function

$$x \to \chi_1 \otimes \psi_1(x\delta) = \chi_1 \otimes \psi_1(\delta^{-1}x\delta) = \chi_1^\delta \otimes \psi_1(x),$$

since δ commutes with H and left multiplication by δ does not change the value of functions in $\mathfrak{L}^2(\Gamma\Delta_0\backslash NH)$. Hence it takes $\chi_1 \otimes \psi_1$ to $\chi_1^\delta \otimes \psi_1$.

Again by [5], $\operatorname{Ind}_{NH\to G}(\chi_1 \otimes \psi_1) = \operatorname{Ind}_{NH\Delta\to G}(\operatorname{Ind}_{NH\to NH\Delta} \chi_1 \otimes \psi_1)$ is irreducible. But from [6], the restriction of $\operatorname{Ind}_{NH\to NH\Delta} \chi_1 \otimes \psi_1$ to NH is just

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$\otimes_{\delta \in NH\backslash NH\Delta} \chi_1^\delta \otimes \psi_1$; furthermore, the action of δ is just to take χ^{δ_1} to $\chi^{\delta_1\delta}$. (This is explicitly worked out in a slightly more general situation in [5], during the proof of Theorem 2.) Thus $\mathrm{Ind}_{NH\to NH\Delta}(\chi_1 \otimes \psi_1)$ is a subrepresentation of the "regular" action of $NH\Delta$ on $\mathcal{L}^2(\Gamma_0\Delta\backslash NH\Delta)$. If we induce this representation to G, we get λ; thus $\mathrm{Ind}_{NH\to G}(\chi_1 \otimes \psi_1)$ is a subrepresentation of λ. As ψ_1 runs through $\Delta_0^\perp$, we get the complete decomposition of the subspace of $\mathcal{L}^2(\Gamma\backslash G) \cong \mathcal{L}^2(\Gamma_0\backslash N) \otimes \mathcal{L}^2(\Delta\backslash K)$ given by $\mathcal{H}_{\chi_1} \otimes \mathcal{L}^2(\Delta\backslash K)$, where $\mathcal{H}_{\chi_1}$ is spanned by the functions χ_1^δ, $\delta \in \Delta$. As χ_1 varies over the Δ-orbits in $\Gamma_0^\perp$, we get the complete decomposition of $\mathcal{L}^2(\Gamma\backslash G)$.

We may summarize this in the following way.

THEOREM 1. $\lambda = \oplus_{\chi,\psi}\mathrm{Ind}_{NH_\chi\to G}(\chi \otimes \psi)$, *where* χ *ranges over the* Δ*-orbits in* Γ_0 *and* ψ *ranges over the characters of* H_χ (= *isotropy subgroup of* χ *in* K) *trivial on* $H_\chi \cap \Gamma$.

(One can also obtain Theorem 1 from the results of [4]; the preceding account is partly an adaptation of the approach employed there to our particular situation.)

COROLLARY. *Suppose that* χ_1 *and* χ_2 *are two elements in* $\Gamma_0^\perp$ *which are in different* Δ*-orbits, but in the same* K*-orbit. Then* λ *is not multiplicity-free.* (*More precisely, the multiplicity of* $\mathrm{Ind}_{NH\to G}\chi_1 \otimes \psi$ *is equal to the number of* K*-orbits of* $\Gamma_0^\perp$ *in the* Δ*-orbit of* χ_1.)

PROOF. Suppose that $\chi_2 = \chi_1^y$. Then $H_{\chi_2} = y^{-1}H_{\chi_1}y = H_{\chi_1} = H$, say, and if $\psi \in \hat{H}$ vanishes on $\Delta \cap H$, then $\mathrm{Ind}_{NH\to G}\chi_1 \otimes \psi$ and $\mathrm{Ind}_{NH\to G}\chi_2 \otimes \psi$ both appear in λ. These two representations are, in fact, equivalent, since $\chi_1 \otimes \psi$ and $\chi_2 \otimes \psi$ are in the same G-orbit. Thus λ contains a representation of multiplicity $\geqq 2$.

Thus, we have the example sought at the beginning of this paper if we can find χ_1 and χ_2 as in the Corollary. (The action of K is then said to violate the Hasse principle.) The rest of this paper is devoted to finding an example of this phenomenon. I am indebted to Professors G. Harder and J. Tits for much of what follows, and especially for their patience in explaining the rudiments of algebraic group theory to me.

Let T' be an Abelian algebraic group/$\mathbf{Q}$ acting on a vector space V; T' will turn out to be an anisotropic torus. Let v be a point in V, and let T be the subgroup of elements fixing v. Then $T'v \cong T'/T$. Galois cohomology gives us an exact sequence (where all groups are taken over $\mathbf{Q}$):

$$T_{\mathbf{Q}} \to T_{\mathbf{Q}}' \to (T'/T)_{\mathbf{Q}} \to H^1(T) \to H^1(T').$$

Suppose that we can find $\xi \in H^1(T)$ such that ξ vanishes locally at every prime (and almost every coboundary is integral), $\xi \neq 0$, and the image of ξ in $H^1(T')$ is 0. Then ξ comes from an element $\alpha \in (T'/T)_{\mathbf{Q}}$. Moreover, α does not have a preimage in T'; otherwise exactness would imply that $\xi = 0$. However, if we work over the adeles, ξ does vanish, and (because of our coboundary condition) α has a preimage in $T_{\mathbf{A}}'$. That is, we have a rational element in $(T'/T)_{\mathbf{A}}$ (or, equivalently, a rational element in $T'_{\mathbf{A}}v$, v rational) which is not the image of a rational element in $T'_{\mathbf{A}}$ (i.e., not in $T'_{\mathbf{Q}}v$). Let

$T'_{\mathbf{A}} = K$, $T_{\mathbf{A}} = H$, $V = N^* = N\hat{}$, and we have the counterexample.

We still need to find T' and T; the existence of an appropriate v is then automatic. (See, e.g., [1, Theorem 5.1 and §5.5].) We let T, T' be the elements of norm 1 of appropriate algebraic number fields k, k', with k' extending k. Let $U = k^\times$ (as an algebraic group/$\mathbf{Q}$); let U_L = group of L-rational points of U, etc. Since $1 \to T_L \to U_L \to L^\times \to 1$ (L an algebraic extension of $\mathbf{Q}$) is an exact sequence of algebraic groups, the usual cohomology exact sequence implies that

$$T_{\mathbf{Q}} \to U_{\mathbf{Q}} \xrightarrow{N} \mathbf{Q}^\times \to H^1(T) \to H^1(U) = 0$$

is exact (N = norm map) and, hence, that $H^1(T) = \mathbf{Q}^\times/N(k^\times)$. Similarly, $H^1(T') = \mathbf{Q}^\times/N(k')$, and it is not hard to check that the inclusion map of T into T' induces $i^*: H^1(T) \to H^1(T')$, given by $i^*(y) = y^n$, where $n = [k': k]$ and $y \in \mathbf{Q}^\times/N(k')$.

The conditions, then, which suffice to give a counterexample are:

(1) $\exists$ an element $y \in \mathbf{Q}^\times$ which is a local norm of k at every prime, but not a global norm;

(2) y^n is a global norm of k' (where $[k' : k] = n$).

This is not hard to arrange. For instance, let $k = \mathbf{Q}(\sqrt{13}, \sqrt{17})$; then [3, p. 360] -1 is a local norm everywhere, but not a global norm. Let $k' = k(\sqrt{-1})$, say. More generally, let k be any Galois extension of $\mathbf{Q}$ such that $\mathbf{Q}^\times$ has elements which are local norms everywhere but not global norms. There are only a finite number of classes of such elements in $H^1(T)$ (see [8, Chapitre 3, Théorème 7']); let $x_1, \ldots, x_m$ be representatives. We know that $x_1^{n_1}$ is a norm for some integer n_1; say $x_1^{n_1} = N_{k/\mathbf{Q}}y_1$. If k_1 is an extension of k of degree n_1 such that $y_1 \in N_{k_1/k}k_1^\times$, then $x_1^{n_1} \in N_{k_1/\mathbf{Q}}y_1$, and so (2) is satisfied for x_1. By induction, we may pick k' to kill off all of $x_1, \ldots, x_m$.

Note, incidentally, that $T_{\mathbf{A}}/T_{\mathbf{Q}}$ is compact. We need one fact found in [9]: $k^\times$ is discrete and cocompact in $k^1_{\mathbf{A}}$, the ideles of norm 1 over k (Chapter 4, Theorem 4). Now we reason as in Lemma 1. The norm map takes $k^\times$ into the closed discrete set $\mathbf{Q}^\times$, and so $k^\times T_{\mathbf{A}}/T_{\mathbf{A}}$ is closed. But if $T_{\mathbf{A}}/T_{\mathbf{Q}}$ is not compact, then $T_{\mathbf{A}}k^\times/k^\times$ is not compact and, hence, $T_{\mathbf{A}}k^\times$ is not closed in $k^1_{\mathbf{A}}$. But then $T_{\mathbf{A}}k^\times/T_{\mathbf{A}}$ is not closed in $k^1_{\mathbf{A}}/T_{\mathbf{A}}$, a contradiction.

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