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INVERTIBLE COMPOSITION OPERATORS ON $L^2(\lambda)$

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ABSTRACT. Let C_ϕ be a composition operator on $L^2(\lambda)$, where λ is a σ-finite measure defined on the Borel subsets of a standard Borel space. In this paper a necessary and sufficient condition for the invertibility of C_ϕ is given in terms of invertibility of ϕ. Also all invertible composition operators on $L^2(\mathbf{R})$ induced by monotone continuous functions are characterised.

Introduction. Let $(X, \mathbb{S}, \lambda)$ be a σ-finite measure space, and let ϕ be a measurable transformation from X into itself. Let $L^2(\lambda)$ denote the Hilbert space of all square-integrable functions on X. Define a composition transformation C_ϕ on $L^2(\lambda)$ as

$$C_\phi f = f \circ \phi \quad \text{for every } f \in L^2(\lambda).$$

In case C_ϕ is a bounded operator with the range in $L^2(\lambda)$, we call it a composition operator induced by ϕ. The Banach space of all bounded linear operators will be denoted by $\mathfrak{B}$. The purpose of this paper is to study the invertible composition operators.

The invertible composition operators on H^p (of the unit disk) are studied by Schwartz [5], where he proves that the invertibility of the inducing function on the unit disk into itself is a necessary and sufficient condition for the invertibility of the corresponding composition operator. This is true because the inducing functions are analytic functions and hence they are nicely behaved. In case of composition operators on $L^2(\lambda)$ the above statement is not completely true as is shown later by an example. However, a little more addition to the hypothesis makes it go through both ways in case of $L^2(\lambda)$.

2. An invertibility theorem.

DEFINITION. Let $(X, \mathbb{S}, \lambda)$ be a measure space. Let ϕ be a measurable transformation on X into itself. Then ϕ is said to be one-to-one if there exists a measurable transformation ψ on X into itself such that $(\psi \circ \phi)(x) = x$ a.e. ϕ is said to be onto if there exists a measurable transformation ω such that $(\phi \circ \omega)(x) = x$ a.e. ϕ is said to be invertible if there is a measurable transformation ψ such that $(\phi \circ \psi)(x) = (\psi \circ \phi)(x) = x$ a.e. Such ψ is called the inverse of ϕ and is denoted as ϕ^{-1}.

DEFINITION. A standard Borel space X is a Borel subset of a complete separable metric space T. The class $\mathbb{S}$ will consist of all the sets of the form $X \cap B$, where B is a Borel subset of T.

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From now on in this section we shall assume that X is a standard Borel space and λ is a σ-finite measure on $\mathbb{S}$. Such kinds of spaces are plentiful and they are mostly used in analysis.

Now suppose ϕ is a measurable transformation from X (standard Borel space) into itself such that $C_\phi \in \mathfrak{B}$. Then we have proved in [4] that ϕ is one-to-one only if the range of C_ϕ is dense in $L^2(\lambda)$. We shall present in the following theorem an analogous result interchanging ϕ and C_ϕ.

THEOREM 1. *Let ϕ be a measurable transformation on X into X such that $C_\phi \in \mathfrak{B}$. Then C_ϕ is one-to-one implies that ϕ is onto.*

PROOF. Suppose C_ϕ is a one-to-one composition operator. Then we need to produce a measurable transformation ω such that $(\phi \circ \omega)(x) = x$ a.e. By Corollary 8.2 of [7], there exist two Borel sets Y and Z such that $Y \subseteq X$, $Z \subseteq \phi(X)$, $\lambda\phi^{-1}(X \backslash Z) = 0$ and ϕ is one-to-one on Y and maps Y onto Z. Since C_ϕ is one-to-one, $\lambda(X \backslash Z) = 0$, otherwise the kernel of C_ϕ will be nontrivial. Define the function ω as $\omega = (\phi/Y)^{-1}$. By Kuratowski's Theorem [2, p. 22] ω is a measurable transformation, and we have $(\phi \circ \omega)(x) = x$ a.e. This shows ϕ is onto.

The converse of the above theorem is not true. We shall cite the following example.

EXAMPLE. Let X be the unit interval $[0, 1]$ with Lebesgue measure λ. Let C be the Cantor set and ψ be the Cantor function. Let ϕ be the function defined by

$$\phi(x) = \tfrac{1}{2}x + \tfrac{1}{2}\psi(x) \quad \text{for every } x \in X.$$

The function ϕ is monotone continuous with $\phi'(x) = \frac{1}{2}$ (ϕ' denotes the derivative of ϕ). It does define a composition operator on $L^2(0, 1)$. If C' denotes the complement of C in X, then $\lambda(\phi(C')) = \frac{1}{2}$, and also $\lambda(\phi(C)) = \frac{1}{2}$. The function $X_{\phi(C)}$, the characteristic function of $\phi(C)$ is in the kernel of C_ϕ because

$$C_\phi X_{\phi(C)} = X_{\phi(C)} \circ \phi = X^{-1}_{\phi\circ\phi(C)} = X_C = 0.$$

Hence C_ϕ is not one-to-one.

Now we shall prove the main theorem of this section.

THEOREM 2. *A composition operator C_ϕ on $L^2(\lambda)$ is invertible if and only if ϕ is invertible and the inverse of ϕ induces a composition operator on $L^2(\lambda)$.*

PROOF. Suppose ϕ is invertible with ψ as its inverse. Suppose $C_\psi \in \mathfrak{B}$. Then $C_\phi C_\psi = C_{\psi\circ\phi} = I = C_{\phi\circ\psi} = C_\psi C_\phi$, where I denotes the identity operator. Hence C_ϕ is invertible with C_ψ as its two sided inverse.

Conversely, suppose C_ϕ is invertible. Let ω be as in the proof of Theorem 1, i.e. $\omega = (\phi/Y)^{-1}$ and $(\phi \circ \omega)(x) = x$ a.e. Since C_ϕ is onto, C_ω is well defined, and $C_\omega = C_\phi^{-1}$. Now in order to complete the proof, we shall show that $(\omega \circ \phi)(x) = x$ a.e. Clearly $(\omega \circ \phi)(x) = x$ for every $x \in Y$. We shall show that $\lambda(X \backslash Y) = 0$. Suppose not. Then $0 < \lambda(X \backslash Y) \leqslant \infty$. If $\lambda(X \backslash Y) < \infty$, then the characteristic function of $X \backslash Y$ is in the kernel of C_ω which is a contradiction to the invertibility of C_ϕ. If $\lambda(X \backslash Y) = \infty$, then we can choose a subset E of $X \backslash Y$ such that $0 < \lambda(E) < \infty$ and the characteristic function of E will be in the kernel of C_ω which is again a contradiction. Therefore $\lambda(X \backslash Y) = 0$. Hence ω is the inverse of ϕ. This completes the proof of the theorem.

COROLLARY. *If C_ϕ is invertible, then $C_\phi^{-1} = C_{\phi^{-1}}$.*

THEOREM 3. *Let ϕ be a real valued monotone continuous function on the set of real numbers $\mathbf{R}$ and let C_ϕ be a bounded composition operator on $L^2(\mu)$, where μ is the Lebesgue measure on the Borel subsets of $\mathbf{R}$. Then C_ϕ is invertible if and only if ϕ' (the derivative of ϕ) is essentially bounded and ϕ is absolutely continuous on the finite intervals.*

PROOF. Suppose C_ϕ is invertible. Then by the theorem, ϕ is invertible and $C_{\phi^{-1}}$ is a bounded operator on $L^2(\mu)$. Hence the Lebesgue-Stieltjes measure induced by ϕ is absolutely continuous with respect to μ, and therefore ϕ is absolutely continuous on finite intervals and $\phi' d\mu = d\phi$ [1, Theorem 19.53]. From this we get

$$\|\phi'\|_\infty = \|d\phi/d\mu\|_\infty = \|C_{\phi^{-1}}\|^2 < \infty.$$

Hence ϕ' is essentially bounded.

On the other hand, if ϕ' is essentially bounded and ϕ is absolutely continuous on finite intervals, then for every continuous function f with the compact support

$$\|C_{\phi^{-1}} f\|^2 = \int |f|^2 \phi' \, d\mu \leqslant \|\phi'\|_\infty \|f\|^2.$$

This is enough to show that $C_{\phi^{-1}}$ is bounded and hence C_ϕ is invertible. This proves the theorem.

COROLLARY. *Let p: $\mathbf{R} \to \mathbf{R}$ be a polynomial such that C_p is bounded. Then C_p is invertible iff the degree of p is equal to* 1.

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