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TWO COUNTEREXAMPLES IN SEMIGROUP THEORY ON HILBERT SPACE[1]

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ABSTRACT. There exist (C_0) semigroups $T_1(t)$, $T_2(t)$ on Hilbert space with the following properties: T_1 has a bounded generator and is uniformly bounded, but is not similar to a contraction semigroup. T_2 is uniformly bounded, and there exists no scalar α such that $e^{-\alpha t}T_2(t)$ is similar to a contraction semigroup.

1. **Introduction.** If $T(t) = e^{tA}$ is a (C_0) semigroup on a Banach space X, then there are real constants $M \geqslant 1$ and β such that $\|T(t)\| \leqslant Me^{\beta t}$. If $\beta = 0$ the semigroup is said to be uniformly bounded; if, in addition, $M = 1$ it is said to be contractive; while if $M = 1$ but $\beta \neq 0$ the semigroup is said to be quasi-contractive. Clearly A generates a quasi-contractive semigroup if and only if there exists a real β such that $A - \beta I$ generates a contractive semigroup, namely $e^{-\beta t}T(t)$. If $T(t)$ is a uniformly bounded semigroup, W. Feller observed that the space X can be renormed to make $T(t)$ contractive; one defines the new norm by $|x| = \sup_{t \geqslant 0}\|T(t)x\|$. Quite generally one can always renorm X by a similar device to make any given (C_0) semigroup quasi-contractive. However, if X is a Hilbert space, the new norm will usually not be a Hilbert norm. Indeed Packel [5] has given an example of a uniformly bounded semigroup $S(t) = e^{tA}$ on Hilbert space H such that there is no equivalent inner product on H which makes $S(t)$ contractive. Equivalently, $S(t)$ is not similar to a contraction semigroup: there is no bounded invertible operator C on H such that $CS(t)C^{-1}$ is a contraction semigroup. The generator A of Packel's semigroup is unbounded, and he asked whether there is an example of such a semigroup with a bounded generator. In §2 we shall present such an example. (We note that Kreiss [4] proved that this phenomenon cannot occur in finite dimensions.)

Goldstein [2], [3] has raised a related question: If $T(t)$ is a (C_0) semigroup on Hilbert space H, is there an α such that the semigroup $e^{-\alpha t}T(t)$ is similar to a contraction semigroup on H? In other words, can H be endowed with an equivalent inner product which makes $T(t)$ quasi-contractive? Goldstein's opinion was that the answer is *no* in general, and in §3 we shall give an example of a semigroup which verifies this conjecture. (The generator of such a semigroup must be unbounded, since if B is bounded we have $\|e^{tB}\| \leqslant e^{t\|B\|}$, so e^{tB} is quasi-contractive.)

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2. In this section we exhibit a bounded operator B on a Hilbert space such that the semigroup e^{tB} is uniformly bounded but not similar to a contractive semigroup.

Let $S(t) = e^{tA}$ be Packel's semigroup on the Hilbert space H. Then $\|S(t)\| \leqslant M$ for all $t \geqslant 0$ and $S(t)$ is not similar to a contraction semigroup on H. For each positive integer n define $A_n = A(I - A/n)^{-1}$; then A_n is a bounded operator and $\|e^{tA_n}\| \leqslant M$ for $t \geqslant 0$ (cf. the proof of the Hille-Yosida-Phillips theorem in [1, VIII.1.13]). Let $B_n = A_n/\|A_n\|$; then $\|B_n\| = 1$ and $S_n(t) = e^{tB_n}$ is uniformly bounded by M. Finally, let $\mathcal{H} = \sum \oplus H_n$ where each summand H_n is a copy of H, and let $B = B_1 \oplus B_2 \oplus B_3 \oplus \cdots$ on $\mathcal{H}$. Then $\|B\| = 1$ and $\|e^{tB}\| \leqslant M$ as operators on $\mathcal{H}$.

PROPOSITION. *The semigroup e^{tB} is not similar to a contraction semigroup.*

PROOF. Arguing by contradiction, suppose that e^{tB} is similar to a contraction semigroup. Then there exists an inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$, equivalent to the original inner product $(\cdot, \cdot)$, with respect to which e^{tB} is contractive.

Let $\langle \cdot, \cdot \rangle_n$ be the restriction of $\langle \cdot, \cdot \rangle$ to the summand H_n, which we identify with H. Then there is a constant $k > 0$ so that

$$k\langle x, x\rangle_n \leqslant (x, x) \leqslant k^{-1}\langle x, x\rangle_n \tag{1}$$

for all vectors x in H and all n. Now define a new inner product on H by

$$[x, y] = \mathrm{LIM}_n \langle x, y\rangle_n \tag{2}$$

where LIM is a fixed Banach limit. Then inequality (1) holds for $[x, x]$ as well, so that $[\cdot, \cdot]$ is equivalent to the original inner product on H.

Now by assumption e^{tB_n} is contractive with respect to the inner product $\langle \cdot, \cdot \rangle_n$, hence so is e^{tA_n} since A_n is just a positive scalar multiple of B_n. Also, the proof of the Hille-Yosida-Phillips theorem in [1] shows that for all x in $H e^{tA_n}x$ converges to $e^{tA}x = S(t)x$. Accordingly,

$$[S(t)x, S(t)x] = \mathrm{LIM}_n \langle e^{tA_n}x, e^{tA_n}x\rangle_n \leqslant \mathrm{LIM}_n \langle x, x\rangle_n = [x, x].$$

That is, $S(t)$ is contractive with respect to the inner product $[\cdot, \cdot]$, a contradiction. □

3. In this section we present an example of a (C_0) semigroup $T(t)$ on Hilbert space such that for no real α is $e^{-\alpha t}T(t)$ similar to a contraction semigroup. The construction makes use of the same machinery employed in §2.

As in §2, let $S(t)$ be Packel's semigroup on H, and let $\mathcal{H}$ be the direct sum of countably many copies of H. On the space $\mathcal{H}$ define

$$T(t) = S(t) \oplus S(2t) \oplus S(3t) \oplus \cdots. \tag{3}$$

PROPOSITION. *$T(t)$ is a uniformly bounded (C_0) semigroup and there does not exist an α such that $e^{-\alpha t}T(t)$ is similar to a contractive semigroup.*

PROOF. On the contrary, suppose that for some α there is an equivalent inner product $\langle \cdot, \cdot \rangle$ on $\mathcal{H}$ with respect to which $e^{-\alpha t}T(t)$ is contractive. As in §2, let $\langle \cdot, \cdot \rangle_n$ be the restriction of $\langle \cdot, \cdot \rangle$ to the nth summand H. Then inequalities (1) hold, and we define a new inner product $[\cdot, \cdot]$ on H by a Banach limit (2) as before.

Now $e^{-\alpha t}S(nt)$ is contractive with respect to $\langle\cdot, \cdot\rangle_n$. If we replace t by t/n it follows that $e^{-\alpha t/n}S(t)$ is also contractive with respect to $\langle\cdot, \cdot\rangle_n$. Applying LIM we deduce that $S(t)$ is contractive with respect to the inner product $[\cdot, \cdot]$, which is again a contradiction. □

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