ON TOTALLY REAL BISECTIONAL CURVATURE

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Abstract. A Kaehler manifold of dimension $\geq 3$ is a complex space form if and only if it has constant totally real bisectional curvature.

Goldberg and Kobayashi [2] introduced the notion of holomorphic bisectional curvature on a Kaehler manifold. It is determined by two holomorphic planes. In this paper, we consider a Kaehler manifold with constant totally real (or called antiholomorphic) bisectional curvature, which is determined by an antiholomorphic plane and its image by the complex structure. Namely it is defined by $R(X,JX; Y,JY)$ for a totally real section $\{X, Y\}$. A complex space form is a Kaehler manifold of constant holomorphic sectional curvature. It turns out that a complex space form can be characterized by having constant totally real bisectional curvature.

1. Let $M$ be a real $2n$-dimensional Kaehler manifold with complex structure $J$ and Riemann metric $g$. Let $R$ be the curvature tensor field of $M$. Then we have $R(JX,JY) = R(X,Y)$ and $R(X,Y)JZ = JR(X,Y)Z$ for any vectors $X$, $Y$, $Z$ tangent to $M$. We denote $R(X,Y;Z,W)$ by

$$R(X,Y;Z,W) = g(R(X,Y)Z,W).$$

Then the sectional curvature of $M$ determined by orthonormal vectors $X$ and $Y$ is given by $K(X,Y) = R(X,Y;Z,W)$. It is easy to see then

$$K(JX,JY) = K(X,Y), \quad K(X,JY) = K(JX,Y)$$

and

$$R(X,Y;Z,W) = R(JX,JY;Z,W) = R(X,Y;JZ,JW).$$

By a plane section we mean a 2-dimensional linear subspace of a tangent space. A plane section $\pi$ is called holomorphic (respectively antiholomorphic or totally real) if $J\pi = \pi$ (respectively $J\pi$ is perpendicular to $\pi$). The sectional curvature for a holomorphic (respectively totally real) plane section is called holomorphic (respectively totally real) sectional curvature. A Kaehler manifold of constant holomorphic sectional curvature is called a complex space form. Let $X$ be a unit vector in a holomorphic plane section, then it is clear

Received by the editors December 17, 1974 and, in revised form, March 3, 1975.


Key words and phrases. Totally real sectional curvature, totally real bisectional curvature, complex space form.

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that the holomorphic sectional curvature of that holomorphic plane section is $K(X,JX)$. We denote $H(X) = K(X,JX)$.

It is easy to see that orthonormal vectors $X$ and $Y$ span a totally real section if and only if $X$, $Y$ and $JY$ are orthonormal. Hence if we use $\{X,Y\}$ to denote the plane section spanned by orthonormal vectors $X$ and $Y$, then that $\{X,Y\}$ is totally real implies that each of $\{X,JY\}$ and $\{Y,JX\}$ is also totally real.

We define the *totally real bisectional curvature* $H(X,Y)$ for a totally real section $\{X, Y\}$ by

$$H(X,Y) = -R(X,JX; Y,JY), \quad \{X, Y\} \text{ is totally real.}$$

We say $M$ is of constant totally real bisectional curvature if $H(X,Y) = \text{constant}$ for every totally real plane section $\{X,Y\}$.

We shall prove the following:

**Theorem.** A Kaehler manifold $M$ with dim $M \geq 3$ is a complex space form if and only if $M$ is of constant totally real bisectional curvature.

2. By the first Bianchi identity we have

$$H(X,Y) = -R(X,JX; Y,JY) = R(X,Y; JY,JX) + R(X,JY; JX,Y) = R(X,Y; Y,X) + R(X,JY; JY,X) = K(X,Y) + K(X,JY).$$

Hence if $M$ is of constant totally real sectional curvature then $M$ is of constant totally real bisectional curvature.

Conversely we assume that $H(X,Y) = -R(X,JX; Y,JY) = C$, $C$ being a constant. Then

$$(2.1) \quad C = K(X,Y) + K(X,JY).$$

Since $\{X, Y\}$ is totally real, the plane section $\{\sqrt{2}^{-1}(X+Y), \sqrt{2}^{-1}(JX-JY)\}$ is also totally real. Thus we have

$$\frac{1}{4}R(X+Y,JX+JY;JX-JY,-X+Y) = -C.$$

After expanding the left side of the above relation and some cancellation we have

$$\frac{1}{4}R(X+Y,JX+JY;JX-JY,-X+Y) = \frac{1}{4}(R(X,JX;X,JX) + R(Y,JY;Y,JY) + 2R(X,JX;Y,JY) - 4R(X,JY;X,JY)) = \frac{1}{4}(-H(X) - H(Y) - 2C + 4K(X,JY)).$$

And hence

$$(2.2) \quad H(X) + H(Y) - 4K(X,JY) = 2C.$$

We can replace $Y$ by $JY$ in (2.2). Noticing that $H(JY) = H(Y)$ we have

$$(2.3) \quad H(X) + H(Y) - 4K(X,Y) = 2C.$$
From (2.1), (2.2) and (2.3) we conclude that

$$H(X) + H(Y) = 4C, \quad K(X, Y) = K(X, JY) = \frac{1}{2}C.$$ 

Hence $M$ has constant totally real sectional curvature. Thus $M$ is of constant totally real bisectional curvature if and only if $M$ is of constant totally real sectional curvature. On the other hand, Chen and Ogiue in [1] proved that if $\dim M > 3$, $M$ has constant totally real sectional curvature if and only if $M$ is a complex space form. The theorem is thus proved.

REFERENCES


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