PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 56, April 1976

ON THE NONEXISTENCE OF GROUPS WITH EXTRA-SPECIAL COMMUTATOR SUBGROUP

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ABSTRACT. In this paper, we extend a result of Joseph and Finkelstein and show that there is no group G such that G' is an extra-special p-group of exponent $> p$ (p odd).

K. Joseph and L. Finkelstein [2] have shown that if p is an odd prime, there does not exist a finite group G satisfying the following three conditions:

(i) G' is an extra-special p-group of exponent $> p$.

(ii) $Z(G) \subseteq G'$.

(iii) G acts irreducibly on $G'/Z(G')$.

It is the object of this paper to prove that their result remains valid even if conditions (ii) and (iii) are dropped. That is, there is no finite group G such that G' is an extra-special p-group of exponent $> p$ (p odd).

Recall that a finite p-group G is called extra-special if $Z(G) = G'$, and $|G'| = p$. We now list a series of lemmas which will be needed for the main theorem. In all that follows, p is an odd prime.

LEMMA 1. *Let G be an extra-special p-group. Then*

(a) $(xy)^p = x^p y^p$,

(b) $x^p \in Z(G)$ *for all* $x, y \in G$.

PROOF. See [1, p. 183].

LEMMA 2. *Let G be an extra-special p-group of exponent $> p$, and let $U = \{x \in G | x^p = 1\}$. Then U is a characteristic subgroup of G and $[G : U] = p$.*

PROOF. The fact that U is a characteristic subgroup follows immediately from Lemma 1(a), and the fact that automorphisms preserve order. The map $x \to x^p$ is a homomorphism of G onto $Z(G)$ with kernel U. Hence $G/U \cong Z(G)$ and $[G : U] = p$.

LEMMA 3. *Let G be a finite p-group of linear transformations acting on a vector space V over a field F of characteristic p. Then some nonzero vector of V is fixed by every element of G.*

PROOF. See [1, p. 31].

LEMMA 4. *Suppose an abelian group G acts as a group of linear transformations*

Received by the editors August 15, 1975.

AMS (MOS) subject classifications (1970). Primary 20F35.

on a vector space V over a field F. Let S be a subspace of V, and H a subgroup of G whose elements induce scalar transformations on S. Let S^G be the subspace of V generated by all vectors s^g, $s \in S$, $g \in G$. Then H also induces scalar transformations on S^G.

PROOF. Let $s_1^{g_1}, s_2^{g_2} \in S^G$, and suppose $h \in H$ with $s^h = \lambda s$ for all $s \in S$. Then

$$\begin{aligned}(s_1^{g_1} + s_2^{g_2})^h &= s_1^{g_1 h} + s_2^{g_2 h} = (s_1^h)^{g_1} + (s_2^h)^{g_2} = (\lambda s_1)^{g_1} + (\lambda s_2)^{g_2} \\ &= \lambda(s_1^{g_1} + s_2^{g_2}).\end{aligned}$$

Similarly, if $c \in F$, then

$$(cs_1^{g_1})^h = c(s_1^{g_1 h}) = c(s_1^h)^{g_1} = \lambda(cs_1^{g_1}).$$

The lemma follows.

THEOREM. *Let G be an extra-special p-group $(p > 2)$ of exponent $> p$. Then there is no finite group K such that $K' = G$.*

PROOF. Suppose $K' = G$. Then K acts by conjugation on G'. Moreover, K acts in a natural way on $G/G' = \overline{G}$; namely, if $k \in K$, and $aG' \in \overline{G}$, then $(aG')^k = (k^{-1}ak)G'$. This is easily seen to be well defined. Now $\tilde{K} = K/G$ is abelian, so we can write $\tilde{K} = \tilde{K}_p \times \tilde{K}_{p'}$, where $\tilde{K}_p$ is a p-group, and $\tilde{K}_{p'}$ has order prime to p. The group $\tilde{K}$ acts in a natural way on $\overline{G}$. Indeed, if $x = kG \in \tilde{K}$, and $aG' \in \overline{G}$, then we define $(aG')^x = (k^{-1}ak)G'$. To see that this is well defined, suppose $x = lG$ and $aG' = bG'$. We need to show that $(k^{-1}ak)G' = (l^{-1}bl)G'$, or equivalently that $l^{-1}b^{-1}lk^{-1}ak \in G'$. But

$$l^{-1}b^{-1}lk^{-1}ak = l^{-1}b^{-1}[kl^{-1}, a^{-1}](ab^{-1})bl \in G',$$

since $kl^{-1} \in G$, and $ab^{-1} \in G'$.

Now $\overline{G}$ is elementary abelian, so it is a vector space over $\mathbf{Z}_p$. Let $U = \{x \in G \mid x^p = 1\}$, and define $\overline{U} = U/G'$. By Lemma 2, U has index p in G, and $\overline{U}$ is a subspace of $\overline{G}$. As U is characteristic in G, $\overline{U}$ is $\tilde{K}_{p'}$-invariant, so by Maschke's Theorem [1, p. 66], there exists a $\tilde{K}_{p'}$-invariant subspace $\overline{W} \subseteq \overline{G}$ such that $\overline{G} = \overline{U} \oplus \overline{W}$. Let $\overline{W} = W/G'$. As $[G: U] = p$, W must have order p^2. Furthermore, W is cyclic since $W \nsubseteq U$.

The action of $\tilde{K}$ on G' is given by a character λ: $\tilde{K} \to \mathbf{Z}$; that is, if $k \in \tilde{K}$, then $c^k = c^{\lambda(k)}$ for all $c \in G'$. Clearly then, $\tilde{K}_{p'}$ acts with character λ on G', hence also on $\overline{W}$. Moreover, $\overline{W}$ is not $\tilde{K}$-invariant. For suppose it were. Then W would be normal in K, and $N_K(W)/C_K(W) = K/C_K(W)$ would be abelian, which implies that $W \subseteq Z(G)$, a contradiction. Thus $\overline{W}^{\tilde{K}} \cap \overline{U} \neq \{1\}$, where $\overline{W}^{\tilde{K}} = \langle w^k \mid w \in \overline{W}, k \in \tilde{K} \rangle$. The p-group $\tilde{K}_p$ acts on $\overline{W}^{\tilde{K}} \cap \overline{U}$, so by Lemma 3, there is a subgroup $\overline{V} \subseteq \overline{W}^{\tilde{K}} \cap \overline{U}$, of order p, which is elementwise fixed by $\tilde{K}_p$.

As $\tilde{K}_{p'}$ acts with character λ on $\overline{W}$, by Lemma 4, it acts with character λ on $\overline{W}^{\tilde{K}}$, in particular on $\overline{V}$. If $\overline{V} = V/G'$, then V is elementary abelian of order p^2, and since $\overline{V}$ is $\tilde{K}$-invariant, $V \lhd K$.

Let K_p and $K_{p'}$ be defined by $K_p/G = \tilde{K}_p$ and $K_{p'}/G = \tilde{K}_{p'}$. Since $\tilde{K}_p$ fixes

$\bar{V}$ elementwise, and K_p fixes G' elementwise, K_p must act on V via matrices of the form $\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}$. Also, the above discussion shows that $K_{p'}$ acts on V via matrices of the form $\begin{pmatrix} \lambda & 0 \\ * & \lambda \end{pmatrix}$.

We conclude that K acts on V via an abelian group of matrices, since matrices of the form $\begin{pmatrix} \mu & 0 \\ * & \mu \end{pmatrix}$ commute. As $K' = G$, G acts trivially on V, that is $V \subseteq Z(G)$. This is a contradiction, and the theorem follows.

It might be worthwhile to note that if a group G is the commutator of any group K, then G is also the commutator of a finite group [3]. Hence the theorem is true without the restriction that K be finite.

ACKNOWLEDGMENTS. The author would like to thank Professor D. S. Passman for several valuable conversations.

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