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SOME INEQUALITIES FOR POLYNOMIALS

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ABSTRACT. Let $p_n(z)$ be a polynomial of degree n. Given that $p_n(z)$ has a zero on the circle $|z| = \rho (0 < \rho < \infty)$ we estimate $\max_{|z|=R>1}|p_n(z)|$ in terms of $\max_{|z|=1}|p_n(z)|$. We also consider some other related problems.

It is well known (see [8, p. 346], or [6, vol. 1, p. 137, Problem III 269]) that if $p_n(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n such that $|p_n(z)| \leqq M$ for $|z| \leqq 1$, then at a point z outside the unit disk

$$(1) \qquad |p_n(z)| \leqq M|z|^n,$$

where equality holds at some point z_0 with $|z_0| > 1$ only if it holds at all such points and that is possible only when $p_n(z) = a_n z^n = Me^{i\gamma} z^n$, i.e. when all the zeros of $p_n(z)$ lie at the origin. It is therefore natural to ask what improvement results from supposing that $p_n(z)$ has a zero of modulus ρ. We have recently proved that in case $\rho = 1$, we may replace (1) by ([4], see (1.7″))

$$(2) \qquad \max_{|z|=R>1}|p_n(z)| \leqq MR^n\left\{1 - \frac{2-\sqrt{2}}{2n}(1 - R^{-1})^2\right\}.$$

The proof of (2) depended very much on the fact that the prescribed zero was located on $|z| = 1$, and could not be modified in any obvious way to deal with the problem in its full generality. Here we prove:

THEOREM 1. *Let $p_n(z)$ be a polynomial of degree n having a zero of modulus ρ $(0 < \rho < \infty)$, and satisfying $|p_n(z)| \leqq M$ for $|z| \leqq 1$. Denote by σ_n and τ_n the smallest positive roots of the equations*

$$(3) \qquad x^{n+1} - 2x + 1 = 0,$$

and

$$(4) \qquad (n+1)x^{n+2} - (n+3)x^{n+1} + (n+1)x - (n-1) = 0,$$

respectively. Then

$$(5) \qquad \begin{aligned} \max_{|z|=R>1}|p_n(z)| &\leqq \frac{d(\rho)R + M}{MR + d(\rho)} MR^n \\ &\leqq d(\rho)R^n + \frac{M^2 - (d(\rho))^2}{M} R^{n-1} \end{aligned}$$

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where

$$(6)\qquad d(\rho) = \begin{cases} 1 - \dfrac{1-\rho}{1+\rho}\dfrac{\rho^n}{1-\rho^n} & \text{if } 0 \leqq \rho \leqq \sigma_n, \\ 1 - \dfrac{1-\rho}{1+\rho}\dfrac{2\rho^{n+1}}{1-\rho^{n+1}} & \text{if } \sigma_n \leqq \rho \leqq \tau_n, \\ \dfrac{n}{n+1}\dfrac{2}{1+\rho} & \text{if } \tau_n \leqq \rho \leqq 1, \\ \dfrac{1}{\rho} d\left(\dfrac{1}{\rho}\right) & \text{if } 1 \leqq \rho. \end{cases}$$

It may be noted that (5) not only extends but also refines (2).

Theorem 1 is an immediate consequence of Lemmas 1 and 2 below.

LEMMA 1. *If $p_n(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n having a zero of modulus ρ then*

$$(7)\qquad |a_n| \leqq d(\rho) \max_{|z|=1} |p_n(z)|$$

where $d(\rho)$ is given by (6). *For small as well as large values of ρ the inequality is essentially best possible.*

Lemma 1 is readily obtained on applying the following result [7, Theorem 3] to the polynomial $z^n p(1/z)$.

THEOREM A. *Let $p_n(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n having a zero of modulus ρ. If σ_n and τ_n denote the smallest positive roots of* (3) *and* (4) *respectively, then*

$$|a_0| \leqq c(\rho) \max_{|z|=1} |p_n(z)|$$

where

$$c(\rho) = \begin{cases} \rho - \dfrac{1-\rho}{1+\rho}\dfrac{\rho^{n+1}}{1-\rho^n} & \text{if } 0 \leqq \rho \leqq \sigma_n, \\ \rho - \dfrac{1-\rho}{1+\rho}\dfrac{2\rho^{n+2}}{1-\rho^{n+1}} & \text{if } \sigma_n \leqq \rho \leqq \tau_n, \\ \dfrac{n}{n+1}\dfrac{2\rho}{1+\rho} & \text{if } \tau_n \leqq \rho \leqq 1, \\ \rho c(1/\rho) & \text{if } 1 \leqq \rho. \end{cases}$$

The estimate is essentially best possible for small as well as for large values of ρ.

LEMMA 2. *If $p_n(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n such that $\max_{|z|=1} |p_n(z)| \leqq M$, then for $|z| = R > 1$,*

$$(8)\qquad |p_n(z)| \leqq \frac{|a_n| R + M}{MR + |a_n|} M R^n;$$

a fortiori, we obtain

$$(8') \qquad |p_n(z)| \leqq |a_n|R^n + \frac{M^2 - |a_n|^2}{M} R^{n-1}.$$

PROOF OF LEMMA 2. It is clear that $q(z) \equiv z^n p_n(1/z)$ is also a polynomial. Besides,

$$|q(e^{i\theta})| \equiv |p_n(e^{-i\theta})| \quad \text{for real } \theta.$$

Hence $\max_{|z|=1}|q(z)| = \max_{|z|=1}|p_n(z)| \leqq M$ and by a well-known generalization of Schwarz's lemma (see for example [5, p. 167])

$$|q(z)| \leqq M\frac{M|z| + |q(0)|}{|q(0)||z| + M} = M\frac{M|z| + |a_n|}{|a_n||z| + M} \quad \text{for } |z| < 1.$$

Replacing z by $1/z$ we obtain the desired result.

REMARK. We observe that $|a_n|$, $(M^2 - |a_n|^2)/M$ appearing on the right-hand side of (8′) cannot in general be replaced by smaller numbers. Given $\varepsilon > 0$ we construct polynomials $p_n(z) = \sum_{k=0}^n a_k z^k$ of degree $n > (2/\varepsilon) - 1$ with

$$\max_{|z|=1}|p_n(z)| \leqq M \quad \text{and}$$

$$\max_{|z|=R}|p_n(z)| > |a_n|R^n + \left(\frac{M^2 - |a_n|^2}{M} - \varepsilon\right)R^{n-1} \quad \text{for } R > \frac{M\sqrt{n}}{\varepsilon}.$$

Let $0 < |\alpha| < M$ and consider the function

$$w = f(z) = M\frac{Mz + \alpha}{\bar{\alpha}z + M} = \alpha + \frac{M^2 - |\alpha|^2}{M}z + \sum_{k=2}^{\infty} c_k z^k,$$

which is analytic in $|z| < M/|\alpha|$ and maps the closed unit disk onto the disk $|w| \leqq M$. If

$$\sigma_n(z) = \frac{s_0(z) + s_1(z) + \cdots + s_n(z)}{n + 1}$$

where $s_0(z), s_1(z), \ldots, s_n(z), \ldots$ are the partial sums of the Taylor series of $f(z)$ then [9, p. 236]

$$\max_{|z|=1}|\sigma_n(z)| \leqq M \quad \text{for } n = 0, 1, 2, \ldots.$$

Hence

$$\begin{aligned} p_n(z) &= z^n \sigma_n(1/z) \\ &= \alpha z^n + \frac{M^2 - |\alpha|^2}{M}\frac{n}{n+1}z^{n-1} + \sum_{k=2}^{n}\frac{n - k + 1}{n + 1}c_k z^{n-k} \end{aligned}$$

is a polynomial of degree n with

$$|p_n(z)| \leqq M \quad \text{for } |z| = 1.$$

Since

$$|\alpha|^2 + \left(\frac{M^2 - |\alpha|^2}{M}\right)^2 + \sum_{k=2}^{\infty} |c_k|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2 \, d\theta \leqq M^2$$

we have

$$\sum_{k=2}^{n} |c_k|^2 \leqq \frac{|\alpha|}{M}(M^2 - |\alpha|^2)^{1/2} \leqq \frac{M}{2}$$

and therefore

$$\begin{aligned}
\max_{|z|=R} |p_n(z)| &\geqq \max_{|z|=R} \left| \alpha z^n + \frac{M^2 - |\alpha|^2}{M} \frac{n}{n+1} z^{n-1} \right| \\
&\quad - \sum_{k=2}^{n} \frac{n-k+1}{n+1} |c_k| R^{n-k} \\
&\geqq |\alpha| R^n + \frac{M^2 - |\alpha|^2}{M} \frac{n}{n+1} R^{n-1} \\
&\quad - \left(\sum_{k=2}^{n} \left(\frac{n-k+1}{n+1} \right)^2 R^{2n-2k} \right)^{1/2} \left(\sum_{k=2}^{n} |c_k|^2 \right)^{1/2} \\
&> |\alpha| R^n + \left\{ \frac{M^2 - |\alpha|^2}{M} - \left(\frac{1}{n+1} \frac{M^2 - |\alpha|^2}{M} + \frac{M}{2} \sqrt{n} \frac{1}{R} \right) \right\} R^{n-1} \\
&> |\alpha| R^n + \left(\frac{M^2 - |\alpha|^2}{M} - \varepsilon \right) R^{n-1}
\end{aligned}$$

if $n > (2/\varepsilon) - 1$ and $R > (M\sqrt{n})/\varepsilon$.

As mentioned earlier, equality holds in (1) only when the coefficients $a_0, a_1, \ldots, a_{n-1}$ are all zero. From Lemma 2 we deduce

THEOREM 2. *If $p_n(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n such that $\max_{|z|=1} |p_n(z)| \leqq M$ and $\max_{0 \leqq k \leqq n-1} |a_k| = a \; (0 \leqq a \leqq M)$, then for $|z| = R > 1$,*

$$(9) \qquad \begin{aligned} |p_n(z)| &\leqq \frac{(M^2 - Ma)^{1/2} R + M}{MR + (M^2 - Ma)^{1/2}} MR^n \\ &\leqq (M^2 - Ma)^{1/2} R^n + aR^{n-1}. \end{aligned}$$

PROOF. If $f(z) = \sum_{k=0}^{\infty} c_k z^k$ is analytic in $|z| < 1$ where $|f(z)| \leqq M$ then ([5, p. 172], see Exercise 9)

$$(10) \qquad |c_0|^2 + |c_k| M \leqq M^2, \qquad k = 1, 2, \ldots.$$

Applying this result to the function

$$z^n p_n(1/z) = a_n + a_{n-1} z + \cdots + a_{n-k} z^k + \cdots + a_0 z^n$$

we obtain $|a_n| \leqq (M^2 - Ma)^{1/2}$ and then (9) follows from Lemma 2.

A theorem of van der Corput and Visser [3] says that if $p_n(z) = \sum_{k=0}^{n} a_k z^k$ is a polynomial of degree n such that $\max_{|z|=1} |p_n(z)| \leqq M$ and $a_u, a_v \; (u < v)$

are two coefficients such that for no other coefficient $a_w \neq 0$ we have $w \equiv u \bmod(v - u)$, then

$$|a_u| + |a_v| \leqq M.$$

Hence, in particular

$$|a_n| \leqq M - \max_{0 \leqq k \leqq (n+1)/2} |a_k| \tag{11}$$

and as another application of Lemma 2 we obtain

THEOREM 3. *If* $p_n(z) = \sum_{k=0}^n a_k z^k$ *is a polynomial of degree* n *such that* $\max_{|z|=1} |p_n(z)| \leqq M$ *and* $\max_{0 \leqq k \leqq (n+1)/2} |a_k| = b$ $(0 \leqq b \leqq M)$, *then for* $|z| = R > 1$

$$|p_n(z)| \leqq \frac{(M-b)R + M}{MR + (M-b)} MR^n. \tag{12}$$

REMARK. It follows from (11) that if $p_n(z) = \sum_{k=0}^n a_k z^k$ is a polynomial of degree n such that $|a_0| \geqq |a_n|$ then $|a_n| \leqq \frac{1}{2} \max_{|z|=1} |p_n(z)|$, and by Lemma 2

$$\begin{aligned} \max_{|z|=R>1} |p_n(z)| &\leqq \frac{(R/2) + 1}{R + (1/2)} R^n \max_{|z|=1} |p_n(z)| \\ &\leqq \left(\tfrac{1}{2} R^n + \tfrac{3}{4} R^{n-1}\right) \max_{|z|=1} |p_n(z)|. \end{aligned} \tag{13}$$

The condition $|a_0| \geqq |a_n|$ is satisfied if for example $p_n(z)$ has all its zeros in $|z| \geqq 1$ or $p_n(z)$ is self reciprocal, i.e. $z^n \overline{p_n(1/\bar{z})} \equiv p_n(z)$. But in these two cases (13) can be replaced by the much stronger (and sharp) inequality ([1], [2])

$$\max_{|z|=R>1} |p_n(z)| \leqq \frac{R^n + 1}{2} \max_{|z|=1} |p_n(z)|. \tag{14}$$

Inequality (13) holds also for polynomials $p_n(z)$ for which $z^n p_n(1/z) \equiv p_n(z)$. However, we do not know the precise estimate in this case. It is readily seen that (14) holds if $n = 1$. We can show that it also holds if $n = 2$. In fact, if $p_2(z) = a_2 z^2 + a_1 z + a_0$ is such that $z^2 p_2(1/z) \equiv p_2(z)$ then $a_2 = a_0$ and

$$\begin{aligned} \frac{\max_{|z|=R>1} |p_2(z)|}{\max_{|z|=1} |p_2(z)|} &= R \frac{\max_{|z|=R>1} |a_2(z + 1/z) + a_1|}{\max_{|z|=1} |a_2(z + 1/z) + a_1|} \\ &= R \max_{w \in \mathfrak{S}} |2a_2 w + a_1| / \max_{w \in [-1,1]} |2a_2 w + a_1| \end{aligned}$$

where $\mathfrak{S}$ is the ellipse with foci at 1, -1 and semiaxes $\frac{1}{2}(R + 1/R)$, $\frac{1}{2}(R - 1/R)$. Hence it is enough to show that for an arbitrary complex number ζ

$$\max_{w \in \mathfrak{S}} |w - \zeta| / \max_{w \in [-1,1]} |w - \zeta| \leqq \frac{1}{2}\left(R + \frac{1}{R}\right).$$

Clearly, there is no loss of generality in assuming that ζ lies in the right half-plane H_1, i.e. $\operatorname{Re} \zeta \geqq 0$. Thus we will like to show that for all ζ lying in H_1,

$$\max_{w \in \mathfrak{S}} \left| \frac{w - \zeta}{1 + \zeta} \right| \leqq \frac{1}{2}\left(R + \frac{1}{R}\right).$$

Now, let $w = u + iv$ be an arbitrary given point on $\mathfrak{S}$. As a function of ζ, $(w - \zeta)/(1 + \zeta)$ is analytic except at the point -1. Hence

$$\max_{\zeta \in H_1} |(w - \zeta)/(1 + \zeta)|$$

cannot be attained at an interior point of H_1. Therefore, all we need to show is that

$$\frac{|u + iv - i\eta|}{\sqrt{(1 + \eta^2)}} \leqq \frac{1}{2}\left(R + \frac{1}{R}\right) \quad \text{for } u + iv \in \mathfrak{S}, \ -\infty < \eta < \infty. \tag{15}$$

But this is a matter of simple verification, and hence

$$\max_{|z|=R>1} |p_2(z)| \leqq \frac{R^2 + 1}{2} \max_{|z|=1} |p_2(z)|$$

if $z^2 p_2(1/z) \equiv p_2(z)$.

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