HOMEO MORPHISMS OF $B^k \times T^n$

TERRY LAWSON

Abstract. An elementary proof is given that a homeomorphism of $B^k \times T^n$ rel $\partial$ that is homotopic to the identity is pseudoisotopic to the identity $\forall k, n$. The key step is to show that each homeomorphism is pseudoisotopic to any standard finite cover.

Let $A_0^{\text{Top}}(B^k \times T^n)$ denote the pseudoisotopy classes of (self-) homeomorphisms of $B^k \times T^n$ rel $\partial$ which are homotopic rel $\partial$ to the identity. We will show $A_0^{\text{Top}}(B^k \times T^n) = 0 \forall k, n$. This should be contrasted with the corresponding PL result, where $A_0^{\text{PL}}(B^k \times T^n) = H^{2-k}(T^n; \mathbb{Z}_2)$ for $n + k \geq 5$ (cf. [2]); in particular, $A_0^{\text{PL}}(B^2 \times T^3) = \mathbb{Z}_2$, and thus $\mathbb{Z}_2$ is responsible for the failure of the Hauptvermutung. That $A_0^{\text{Top}}(B^k \times T^n) = 0$ for $k + n \geq 5$ was known [6] using topological surgery. However, our result is completely elementary, relying only on the results of Lawson [3], the local contractibility of the homeomorphism group of a compact manifold [1], and an easy inductive argument.

We review the main result of [3], rephrased in terms relevant for our application. It says that there is an exact sequence.

$$0 \to A_0^{\text{Top}}(B^{k+1} \times T^{n-1}) \xrightarrow{i} A_0^{\text{Top}}(B^k \times T^n) \xrightarrow{j} IC_0(B^k \times T^{n-1})/SC(B^k \times T^{n-1}) \to 0.$$  

The map $i$ comes from regarding $B^k \times T^n$ as $B^k \times T^{n-1} \times I/B^k \times T^{n-1} \times \{0, 1\}$. The map $j$ takes a homeomorphism $g$ of $B^k \times T^n$, regards it as a homeomorphism of $B^k \times T^{n-1} \times S^1$, lifts it to $\bar{g}: B^k \times T^{n-1} \times \mathbb{R}$ so that $\bar{g}(B^k \times T^{n-1} \times 0)$ lies above $B^k \times T^{n-1} \times 0$, and then $j([\bar{g}])$ is the invertible $B^k \times T^{n-1}$-cobordism $(W, i_0, \bar{g}i_0, I)$, where $W$ is the region between $B^k \times T^{n-1} \times 0$ and $\bar{g}(B^k \times T^{n-1} \times 0)$,

$$i_0: B^k \times T^{n-1} \to B^k \times T^{n-1} \times 0 \subset B^k \times T^{n-1} \times \mathbb{R}$$

is the inclusion and $I$ is a trivialization along the boundary (cf. [3], [4]). The main property of $j$ which we need is that if $\bar{g}$ denotes the $d$-fold cover of $g$ making the following diagram

Received by the editors January 9, 1975.


Key words and phrases. Pseudoisotopy, homeomorphism, PL homeomorphism, Hauptvermutung, invertible cobordism, isotopy.

1 Supported in part by NSF GP-43943.
\[
\begin{array}{c}
\mathcal{B}^k \times T^{n-1} \times \mathbb{R} \xrightarrow{\mathcal{g}} \mathcal{B}^k \times T^{n-1} \times \mathbb{R} \\
\downarrow \hspace{1cm} \downarrow \\
\mathcal{B}^k \times T^{n-1} \times S^1 \xrightarrow{\mathcal{g}} \mathcal{B}^k \times T^{n-1} \times S^1 \\
\downarrow \hspace{1cm} \downarrow \\
\mathcal{B}^k \times T^{n-1} \times S^1 \xrightarrow{\mathcal{g}} \mathcal{B}^k \times T^{n-1} \times S^1
\end{array}
\]

commute, then \( j([\mathcal{g}]) = j([g]) \).

**Theorem.** \( A_{0}^{\text{Top}}(\mathcal{B}^k \times T^n) = 0 \).

**Proof.** We proceed by induction on \( n \). If \( n = 0 \) then the result holds by the Alexander isotopy. Our inductive hypothesis then simplifies the exact sequence to an isomorphism

\[
A_{0}^{\text{Top}}(\mathcal{B}^k \times T^n) \cong IC_{0}(\mathcal{B}^k \times T^{n-1})/SC(\mathcal{B}^k \times T^{n-1}).
\]

Now if \( g \) represents an element of \( A_{0}^{\text{Top}}(\mathcal{B}^k \times T^n) \), and \( \mathcal{g} \) is a \( d \)-fold cover as above, we have \( j([\mathcal{g}]) = j([g]) \), hence \([\mathcal{g}] = [g] \). By interchanging the factors of \( S^1 \) in \( T^n \), this argument then implies that if \( \mathcal{g} \) is any standard finite cover of \( g \), then \([\mathcal{g}] = [g] \). But by [1] and [6, Lemma 4.1], if \( \mathcal{g} \) is the standard \( d^n \)-fold cover for \( d \) large enough, \( \mathcal{g} \) is isotopic (hence pseudoisotopic) to the identity.

**Remark.** The reader may find further applications of the exact sequence of [3], including others of relevance to the failure of the Hauptvermutung, in [3], [4], and [5].

**References**