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A NOTE ON THE EQUATION $x^2 = y^q + 1$

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ABSTRACT. It is proved here that the equation $x^2 = y^q + 1$ has no solution in natural numbers x, y for which q is a prime > 3.

It was shown by Chao Ko [1], [2] that the equation $x^2 = y^q + 1$ has no solution in natural numbers x, y where q is a prime > 3.

It is the purpose of this note to give a simpler proof of Ko Chao's result.

Throughout this note all symbols denote natural numbers, and notation $(a, b) = g$ means g is the greatest common divisor of a and b.

Auxiliary lemmas. Lemmas 1 and 2 are collections of some well-known results. The third is due to Nagell [3].

LEMMA 1. *If q is an odd prime and $(x,y) = 1$, then $x + y$ divides $x^q + y^q$ and $(x + y, (x^q + y^q)/(x + y)) = q$ or 1 according as $x + y$ is divisible by q or not.*

LEMMA 2. *All the primitive solutions of equation $x^2 + y^2 = z^2$ for which y is an even number are given by the formulas*

$$x = a^2 - b^2, \quad y = 2ab \quad \text{and} \quad z = a^2 + b^2 \qquad (a > b).$$

LEMMA 3. *If $x^2 = y^q + 1$ with q prime and $x \geqslant 1, y \geqslant 1$, then $2|y$ and $q|x$.*

Principal result.

THEOREM. *Let q be a prime > 3. The equation $x^2 = y^q + 1$ has no solution in natural numbers.*

PROOF. Let us assume now that there exist x, y and a prime q for which $x^2 = y^q + 1$.

It follows from Lemma 3 that $q|x$ and $2|y$. Since $2 \nmid x$, by Lemma 3 we have $(x + 1, x - 1) = 2$. Thus either

(I) $x + 1 = 2^{q-1}y_1^q, \quad x - 1 = 2y_2^q$, or

(II) $x + 1 = 2y_2^q, \quad x - 1 = 2^{q-1}y_1^q$

holds, where $y = 2y_1y_2$, $2 \nmid y_2$ and $(y_1, y_2) = 1$.

Case I. Suppose $x + 1 = 2^{q-1}y_1^q$ and $x - 1 = 2y_2^q$. It follows from $y_2^q = 2^{q-2}y_1^q - 1$ that

$$(1) \qquad (y_2^2)^q + (2y_1)^q = (y_2^q + 2)^2 = ((x + 3)/2)^2.$$

Since $q|x$ and $q > 3$ we see $q \nmid (x + 3)/2$; thus

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$$(y_2^2 + 2y_1, ((y_2^2)^q + (2y_1)^q)/(y_2^2 + 2y_1)) = 1,$$

by Lemma 3.

In view of (1), it follows that $y_2^2 + 2y_1 = h^2$ where $h|(x + 3)/2$. This gives

$$(2) \qquad (hy_2)^2 + y_1^2 = (y_2^2 + y_1)^2.$$

Since $(y_1, y_2) = 1$, this implies $(hy_2, y_1) = 1$. We observe that since y_2 is odd, so is h; then $4|h^2 - y_2^2$ so that $2|y_1$.

By Lemma 2, the solutions of (2) are given by

$$hy_2 = a^2 - b^2, \quad y_1 = 2ab \quad \text{and} \quad y_2^2 + y_1 = a^2 + b^2 \qquad (a > b).$$

Therefore, $(a - b)^2 = (y_2^2 + y_1) - y_1 = y_2^2$ which implies $y_2 = a - b$, and so $y_1 - y_2 = 2ab - (a - b) = a(2b - 1) + b > 0$, hence $y_1 > y_2$. However, $y_2^q = 2^{q-2}y_1^q - 1 > y_1^q$ implies $y_2 > y_1$, and this is impossible. This completes Case I.

Case II. The proof for this case proceeds similarly. It can be easily seen from $(y_2^2)^q - (2y_1)^q = (y_2^q - 2)^2 = ((x - 3)/2)^2$ follows $y_2^2 - 2y_1 = h^2$ where $h|(x - 3)/2$; this implies $(hy_2)^2 + y_1^2 = (y_2^2 - y_1)^2$ so that Lemma 2 gives

$$hy_2^2 = a^2 - b^2, \quad y_1 = 2ab, \quad y_2^2 - y_1 = a^2 + b^2 \qquad (a > b).$$

Hence $y_1 - y_2 = 2ab - (a + b) = (a - 1)(b - 1) + (ab - 1) > 0$ which is impossible because of $y_2^q = 2^{q-2}y_1^q + 1 > y_1^q$. This completes the proof of our Theorem.

REFERENCES

1. Chao Ko, Acta Sci. Natur. Univ. Szechuan. **2** (1960), 57–64.

2. ———, *On the Diophantine equation* $x^2 = y^n + 1$, $xy \neq 0$, Sci. Sinica **14** (1965), 457–460. MR **32** #1164.

3. T. Nagell, *Sur l'impossibilité de l'equation indéterminée* $y^2 = z^p + 1$, Norsk. Mat. Forenings Skrifter I **4** (1921).

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