WEAK CONVERGENCE OF SEMIGROUPS
IMPLIES STRONG CONVERGENCE OF
WEIGHTED AVERAGES

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Abstract. For a fixed $p$, $1 < p < \infty$, let $\{T_t : t > 0\}$ be a strongly continuous semigroup of positive contractions on $L_p$ of a $\sigma$-finite measure space. We show that weak convergence of $\{T_t : t > 0\}$ in $L_p$ is equivalent with the strong convergence of the weighted averages $\int_0^\infty T_t f \mu_n(dt)$ ($n \to \infty$) for every $f \in L_p$ and every sequence $(\mu_n)$ of signed measures on $(0, \infty)$, satisfying $\sup_n \|\mu_n\| < \infty$; $\lim_n \mu_n(0, \infty) = 1$; and for each $d > 0$, $\limsup_{n \to \infty} |\mu_n|(c, c + d) = 0$. The positivity assumption is not needed if $p = 1$ or $2$. We show that such a result can be deduced—not only in $L_p$, but in general Banach spaces—from the corresponding discrete parameter version of the theorem.

In recent years, various authors have studied the relations between weak and strong operator convergence: Blum-Hanson [4], Hanson-Pledger [7], Lin [9], Akcoglu-Sucheston [1], Jones-Kuftinec [8], Fong-Sucheston [6], and very recently Akcoglu-Sucheston [2], [3], who proved the theorem for positive $L_p$-contractions, $1 < p < \infty$. This theorem [1], [6], [3] states that if $T$ is a positive contraction on $L_p$ of a $\sigma$-finite measure space $(X, \mathcal{E}, \mu)$, where $p$ is fixed and $1 < p < \infty$, then weak-$\lim_{n \to \infty} T^n f \left(\text{w-lim}_{n \to \infty} T^n f \right)$ exists for each $f \in L_p$ if, and only if, $\lim_{n \to \infty} \sum_{m=1}^\infty a_{nm} T^n f$ exists for every $f \in L_p$ and every matrix $(a_{nm})$ with real entries satisfying

$$\sup_n \sum_m |a_{nm}| < \infty; \lim_n \sum_m a_{nm} = 1; \lim \max_n |a_{nm}| = 0.$$ 

(1.1)

It has also been shown that the positivity assumption is not needed if $p = 1$ or 2 [1], [6]. The problem of whether or not positivity is needed for $p \neq 1$ or 2 is still open. Matrices satisfying (1.1) were introduced in ergodic context in [6] and have been called uniformly regular; we denote the class of all uniformly regular matrices by $\mathcal{A}_R$. Intuitively, a matrix $(a_{nm}) \in \mathcal{A}_R$ if and only if it is properly averaging, in the sense that the masses $a_{nm}$ spread as $n \to \infty$.

A semigroup $\{T_t : t > 0\}, T_t T_s = T_{t+s}$, of linear operators on a Banach space $B$ is called strongly continuous if for each $x \in B$ and each $s > 0$, $\lim_{t \to s} \|T_t x - T_s x\| = 0$. R. Sato [10] recently obtained the following continuous parameter version of the strong ergodic theorem: For a fixed function $f$ in $L_2$ and a strongly continuous semigroup $\{T_t : t > 0\}$ of contractions on $L_2$, w-$\lim_{t \to \infty} T_t f = f_0$ implies $\lim_{n \to \infty} \int_0^\infty a_n(t) T_t f dt = f_0$ for every sequence $(a_n)$ of nonnegative, Lebesgue integrable functions on $(0, \infty)$ satisfying

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In this note, we show that a stronger result can be deduced—not only in $L_2$, but in general Banach spaces—from the corresponding discrete version of the theorem ($\S$2). In $\S 3$, we obtain as corollaries the continuous parameter version of the Akcoglu-Sucheston theorem for the Banach spaces $L_p(X,\mathcal{E},m)$, $1 \leq p < \infty$.

A linear operator $T$ on a Banach space $B$ is called \textit{power-bounded} if $\sup_n \|T^n\| < \infty$; a semigroup $\{T_t: t > 0\}$ of linear operators on $B$ is called \textit{uniformly bounded} if $\sup_{t > 0} \|T_t\| < \infty$. The total variation of a signed measure $\mu$ is denoted by $|\mu|$. We denote by $\mathcal{F}$ the family of all sequences $(\mu_n)$ of signed measures on the $\sigma$-algebra of Lebesgue measurable subsets of $(0, \infty)$ satisfying

$$\sup_n \|\mu_n\| < \infty; \quad \lim_{n \to \infty} \mu_n(0, \infty) = 1;$$

(2.1)

$$\lim_{n \to \infty} \sup_{c > 0} |\mu_n|(c, c + d] = 0 \quad \text{for each } d > 0.$$

\textbf{Remark 1.} If $(a_n)$ is a sequence of Lebesgue integrable functions on $(0, \infty)$ satisfying

$$\sup_n \int_0^\infty |a_n(t)| \, dt < \infty; \quad \lim_{t \to \infty} \int_0^\infty a_n(t) \, dt = 1;$$

(2.2)

$$\lim_{n \to \infty} \sup_{c > 0} \int_c^{c+d} |a_n(t)| \, dt = 0$$

for each $d > 0$, and if we set $d\mu_n = a_n \, dt$, then $(\mu_n) \in \mathcal{F}$. We note that sequences $(a_n)$ satisfying (2.2) include those considered by Sato in [10].

\textbf{Remark 2.} Let $\delta(t)$ denote the unit point mass at $t$. If $t_m > 0$, $t_m \to \infty$, $(a_{nm}) \in \mathcal{F}_R$, and if we set $\mu_n = \sum_m a_{nm} \delta(t_m)$, then $(\mu_n) \in \mathcal{F}$.

\textbf{Remark 3.} If $x(t)$ is a bounded continuous function from $(0, \infty)$ to a Banach space and if we set $d\mu_n = a_n \, dt$, then $(\mu_n) \in \mathcal{F}$. We note that sequences $(a_n)$ satisfying (2.2) include those considered by Sato in [10].

\textbf{Theorem 2.1.} Let $x$ be a fixed element in a Banach space $B$, real or complex. Then (a) implies (b):

(a) For every power bounded linear operator $T$ on $B$, if $w\lim_{n \to \infty} T^n x = x_0$, then $\lim_{n \to \infty} \sum_{m=1}^\infty a_{nm} T^m x = x_0$ for every matrix $(a_{nm}) \in \mathcal{F}_R$.

(b) For every uniformly bounded semigroup $\{T_t: t > 0\}$ of linear operators on $B$ for which $T_t x$ is continuous on $(0, \infty)$, if $w\lim_{t \to \infty} T_t x = x_0$, then $\lim_n \int_0^{\infty} T_t x \mu_n(dt) = x_0$ for every sequence $(\mu_n) \in \mathcal{F}$.

The conclusion remains valid if “power bounded” and “uniformly bounded” in (a) and (b) are both replaced by “contraction”.

\textbf{Proof.} Let $x$ be a fixed element in $B$, and assume that (a) holds for $x$. Let $\{T_t: t > 0\}$ be a semigroup satisfying the hypotheses of (b) and $(\mu_n) \in \mathcal{F}$. We shall show that $\lim_n \int_0^{\infty} T_t x \mu_n(dt) = x_0$.

Let $\varepsilon > 0$. The continuity of $T_t x$ on $[1, 2]$ implies that $T_t x$ is uniformly continuous on $[1, 2]$. Thus there is a positive integer $k$ such that if
\[
g(t) = \sum_{j=1}^{k} I_{(1+(j-1)/k,1+j/k)}(t) \cdot T_{i+j/k} x,
\]
then \(\|g(t) - T_t x\| < \varepsilon\) for \(t \in (1, 2]\). Here \(I_A\) denotes the function that is 1 on \(A\), and 0 elsewhere. Set \(M = \sup_{t > 0} \|T_t\|\), \(K = \sup_n \|\mu_n\|\), and \(I_i = (i, i+1]\). Since \(|\mu_n|(I_0) \to 0\) by (2.1), we have

\[
\begin{align*}
\limsup_{n \to \infty} \left\| \int_0^\infty T_t x \mu_n(dt) - \sum_{i=0}^{\infty} \int_{I_{i+1}} T_t g(t-i) \mu_n(dt) \right\| \\
= \limsup_{n \to \infty} \left\| \int_0^\infty T_t x \mu_n(dt) + \sum_{i=0}^{\infty} \int_{I_{i+1}} (T_t x - T_t g(t-i)) \mu_n(dt) \right\| \\
\leq \limsup_{n \to \infty} \left[ M \cdot \|x\| \cdot |\mu_n|(I_0) \\
+ \sum_{i=0}^{\infty} \|T_i\| \cdot \sup_{t \in I_i} \|T_t x - g(t)\| \cdot |\mu_n|(I_{i+1}) \right] \\
\leq M \cdot K \cdot \varepsilon.
\end{align*}
\]

For each \(i > 0\), \(1 \leq j \leq k\), set \(I_{i,j} = (i + (j-1)/k, i + j/k]\). It follows from the definition of \(g(t)\) that for \(n \geq 1\),

\[
\begin{align*}
\sum_{i=0}^{\infty} \int_{I_{i+1}} T_t g(t-i) \mu_n(dt) &= \sum_{i=0}^{\infty} \sum_{j=1}^{k} \mu_n(I_{i+1,j}) \cdot T_{i+1+j/k} x \\
&= \sum_{m=k+1}^{\infty} a_{n,m} T^m x,
\end{align*}
\]

where \(T = T_{1/k}\), and for \(m = (i+1)k + j\), \(0 \leq j \leq k\), \(a_{n,m} = \mu_n(I_{i+1,j})\). It is easily checked that \((a_{n,m}) \in \mathfrak{K}_R\) since \((\mu_n) \in \mathfrak{K}\). Moreover, since \(\{T_t: t > 0\}\) is uniformly bounded and \(\text{w-lim}_{t \to \infty} T_t x = x_0\), we have that \(T\) is power bounded and \(\text{w-lim}_{m \to \infty} T^m x = x_0\). Thus it follows from (a) that \(\lim_{n \to \infty} \sum_{m=k}^{\infty} a_{n,m} T^m x = x_0\). Together with (2.3) and (2.4), we obtain that

\[
\limsup_{n \to \infty} \left\| \int_0^\infty T_t x \mu_n(dt) - x_0 \right\| \leq M K \varepsilon.
\]

As \(\varepsilon > 0\) is arbitrary, (\(\beta\)) holds.

It is clear that the second part of the theorem can be proved in the same way.

\textbf{Corollary 2.1.} Let \(B\) be a Banach space. Then (\(\alpha\)) implies (\(\beta\)):

(\(\alpha\)) For every power bounded linear operator \(T\) on \(B\), if \(\text{w-lim}_{n \to \infty} T^n x\) exists for every \(x \in B\), then \(\lim_{n \to \infty} \sum_{m=k}^{\infty} a_{n,m} T^m x\) exists and is equal to \(\text{w-lim}_{n \to \infty} T^n x\) for every \((a_{n,m}) \in \mathfrak{K}_R\).

(\(\beta\)) For every strongly continuous, uniformly bounded semigroup \(\{T_t: t > 0\}\) of linear operators on \(B\), if \(\text{w-lim}_{t \to \infty} T_t x\) exists for every \(x \in B\), then \(\lim_{n} \int_0^\infty T_t x \mu_n(dt)\) exists for every sequence \((\mu_n) \in \mathfrak{K}\), and is equal to \(\text{w-lim}_{t \to \infty} T_t x\).

The conclusion remains valid if “power bounded” and “uniformly bounded” are both replaced by “contraction”.

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Proof. Immediate from Theorem 2.1. □

We next show that the converse of statement (β) in Theorem 2.1 is valid in general Banach spaces.

Proposition 2.1. Let \(x\) be a fixed element in a Banach space \(B\), real or complex. Let \((T_t: t > 0)\) be continuous linear operators on \(B\) such that the vector-valued function \(T_t x\) from \((0, \infty)\) to \(B\) is continuous and \(\sup_{t>0} \|T_t x\| < \infty\). Then (b) implies (a):

(a) \(\text{w-lim}_{t \to \infty} T_t x\) exists.

(b) \(\lim_{n \to \infty} \int_0^\infty T_t x \mu_n(dt)\) exists for every sequence \((\mu_n) \in \mathcal{F}\).

Proof. We first consider the case where \(B\) is a real Banach space. Assume that (b) holds but (a) fails. Then there exists an \(x^* \in B^*\) such that \(h(t) = \langle T_t x, x^* \rangle\) diverges as \(t \to \infty\), where \(B^*\) is the dual space of \(B\). Since

\[
\sup_{t>0} |h(t)| \leq \sup_{t>0} \|x^*\| \|T_t x\| < \infty,
\]

\(h\) is bounded on \((0, \infty)\). \(h(t)\) is also continuous on \((0, \infty)\) since \(T_t x\) is. Thus, \(h(t)\) being divergent as \(t \to \infty\), there are constants \(\alpha, \beta\) with \(\alpha < \beta\), and a sequence \((\xi_i)\) with \(\xi_i \uparrow \infty\), such that \(h(\xi_i) > \beta\) if \(i\) is odd, and \(h(\xi_i) \leq \alpha\) if \(i\) is even. Set for \(n > 1\),

\[
\mu_{2n} = \frac{1}{n} \sum_{k=1}^{n} \delta(t_{2k}), \quad \mu_{2n-1} = \frac{1}{n} \sum_{k=1}^{n} \delta(t_{2k-1}),
\]

where \(\delta(t)\) denotes the unit point mass at \(t\). Then \((\mu_n) \in \mathcal{F}\), but

\[
\liminf_{n} \left< \int_0^\infty T_t x \mu_n(dt), x^* \right> = \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(t_{2k})
\]

\[
\leq \alpha < \beta \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(t_{2k-1})
\]

\[
= \limsup_{n \to \infty} \left< \int_0^\infty T_t x \mu_n(dt), x^* \right>.
\]

Hence \((\int_0^\infty T_t x \mu_n(dt))_{n=1}^{\infty}\) does not converge weakly, and a fortiori, strongly.

If \(B\) is a complex Banach space, then either the real part or the imaginary part of \(h(t)\) diverges as \(t \to \infty\), and can be used to replace \(h(t)\) in the above argument. □

Remark 4. The vector-valued function \(T_t x\) in Proposition 3.1 may be replaced by any vector-valued function \(x(t)\) from \((0, \infty)\) to \(B\) such that \(x(t)\) is continuous and bounded on \((0, \infty)\).

3. We now apply the results in §2 to the Banach spaces \(L_p\) of a \(\sigma\)-finite measure space \((X, \mathcal{A}, m), 1 \leq p < \infty\). An operator \(T\) on \(L_p\) is called positive if \(Tf \geq 0\) whenever \(f \geq 0\). Theorem 3.1 below strengthens the result of R. Sato mentioned in §1.

Theorem 3.1. Let \((T_t: t > 0)\) be a contraction semigroup on \(L_2(X, \mathcal{A}, m)\), and let \(f\) be a fixed function in \(L_2\) such that \(T_f\) is continuous on \((0, \infty)\). Then conditions (a) and (b) are equivalent:
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(a) $\text{w-lim}_{t \to \infty} T_t f = f_0$.
(b) $\lim_n \int_0^\infty T_t f \mu_n (dt) = f_0$ for every $(\mu_n) \in \mathcal{A}$.

**Proof.** This follows from Theorem 1.1 in [6], Proposition 2.1 and Theorem 2.1. □

**Theorem 3.2.** Let $\{T_t; t > 0\}$ be a strongly continuous contraction semigroup on $L_1(S, \mathcal{A}, m)$. Then conditions (A) and (B) are equivalent:

(A) For each $f \in L_1$, $\text{w-lim}_{t \to \infty} T_t f$ exists.
(B) For each $f \in L_1$, $\lim_n \int_0^\infty T_t f \mu_n (dt)$ exists for every $(\mu_n) \in \mathcal{A}$, and is equal to $\text{w-lim}_{t \to \infty} T_t f$.

**Proof.** This follows from Theorem 1.3 in [6], Proposition 2.1, and Corollary 2.1. □

**Theorem 3.3** Let $\{T_t; t > 0\}$ be a strongly continuous semigroup of positive contractions on $L_p(X, \mathcal{A}, m)$, where $p$ is fixed, $1 < p < \infty$. Then conditions (A) and (B) are equivalent:

(A) For each $f \in L_p$, $\text{w-lim}_{t \to \infty} T_t f$ exists.
(B) For each $f \in L_p$, $\lim_n \int_0^\infty T_t f \mu_n (dt)$ exists for each sequence $(\mu_n) \in \mathcal{A}$, and is equal to $\text{w-lim}_{t \to \infty} T_t f$.

**Proof.** We observe that the conclusions in Theorem 2.1 and Corollary 2.1 remain valid if $B = L_p(X, \mathcal{A}, m)$, and “power bounded” and “uniformly bounded” in (A) and (B) are both replaced by “positive contraction”. Theorem 3.3 now follows from Theorem 1.4 in [3] and Proposition 2.1. □

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**References**


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