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A NOTE ON SOME PROPERTIES OF A- FUNCTIONS

H. SARBADHIKARI

ABSTRACT. This note deals with (**M**, $*$) functions for various families **M**. It is shown that if **M** is the family of Borel sets of additive class α on a metric space X, then (**M**, $*$) functions are just the functions of the form $\sup_y g(x,y)$ where $g: X \times R \to R$ is continuous in y and of class α in x. If **M** is the class of analytic sets in a Polish space X, then the (**M**, $*$) functions dominating a Borel function are just the functions $\sup_y g(x,y)$ where g is a real valued Borel function on X^2. It is also shown that there is an A-function f defined on an uncountable Polish space X and an analytic subset C of the real line such that $f^{-1}(C) \notin$ the σ-algebra generated by the analytic sets on X.

1. **Introduction.** Let X be any set and **M**, **N** be classes of subsets of X. Following Hausdorff, we call a real valued function f on X a function of class (**M**, $*$) if $\{x: f(x) > c\}$ is in **M** for every c. If $\{x: f(x) \geqslant c\}$ is in **N** for every c, f is said to be of class ($*$, **N**). Set (**M**, **N**) $=$ (**M**, $*$) $\cap$ ($*$, **N**).

If X is a metric space and **M** is the family of sets of additive Borel class α, then functions of class (**M**, $*$) are called α^--functions; if X is Polish and **M** is the family of analytic sets, they are called A-functions. We shall prove the following theorems:

THEOREM 1. *Let f be a real valued function on a metric space X. Then f is an α^--function if, and only if, there is a real valued function g defined on $X \times R$, where R is the real line, such that $g(x, y)$ is a continuous function of y for fixed x, is of class α in x for fixed y and $f(x) = \sup_y g(x,y)$.*

THEOREM 2. *Let X be a Polish space and let f be a real valued function on X which is bounded below. Then f is an A-function if, and only if, there is a real valued Borel function g on X^2 such that $f(x) = \sup_y g(x,y)$.*

THEOREM 3. *Let* **A** *be the σ-algebra generated by analytic sets on an uncountable Polish space X. There is an A-function f on X and an analytic subset C of the real line such that $f^{-1}(C) \notin$* **A**.

Theorem 3 answers in the negative a question raised by David Blackwell.

2. **Proof of Theorem 1.** We define a complete ordinary function system on a set X as a system **F** of real valued functions on X satisfying:

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(a) Every constant function is in **F**.

(b) If $f, g \in \mathbf{F}$, then $\max(f,g)$, $\min(f,g)$, $f \pm g$, $f \cdot g \in \mathbf{F}$. If g does not vanish anywhere, then $f/g \in \mathbf{F}$.

(c) If $f_n \in \mathbf{F}$ for all n and f_n converges uniformly to f, then $f \in \mathbf{F}$.

We first prove the following:

THEOREM 4. *Let* **F** *be a complete ordinary function system on a set X. Let* **P**, **Q** *be the families of sets* $\{x\colon h(x) > c\}$, $\{x\colon h(x) \geqslant c\}$, *for* $h \in \mathbf{F}$ *and c real, respectively.* $f \in (\mathbf{P}, *)$ *if, and only if, there is a real valued function g defined on* $X \times R$ *such that* $g(x, y)$

(a) *is continuous in y for fixed x,*

(b) *is in* **F** *for fixed y, and*

(c) $\sup_y g(x,y) = f(x)$.

PROOF. Suppose $g(x,y)$ is a function on $X \times R$ satisfying conditions (a) and (b) and suppose $\sup_y g(x,y)$ exists and is $f(x)$. Let c be any real number. Then $f(x) > c \Leftrightarrow \exists y\{g(x,y) > c\} \Leftrightarrow \exists y\{y$ is rational and $g(x,y) > c\}$, since $g(x, y)$ is continuous in y. Thus

$$\{x\colon f(x) > c\} - \bigcup_{r \text{ rational}} \{x\colon g(x,r) > c\}.$$

For fixed r, $g(x,r) \in \mathbf{F}$ and hence $\{x\colon g(x,r) > c\} \in \mathbf{P}$. Now as **P** is closed under countable unions (cf. [1]), $\{x\colon f(x) > c\} \in \mathbf{P}$.

Conversely, suppose $f \in (\mathbf{P}, *)$. It is shown in [1] that there is an increasing sequence $\{f_n\}$ in **F** which converges to f. Define g on $X \times R$ by $g(x,y) = (f_{n+1}(x) - f_n(x))(|y| - n) + f_n(x)$ for $|y| \in [n, n+1]$. It is easy to see that g is well defined for all (x, y) and satisfies (a) and (b). As $f_n(x) \leqslant g(x,y) \leqslant f_{n+1}(x)$ for $|y| \in [n, n+1]$ and $\sup_n f_n(x) = f(x)$, $\sup_y g(x,y) = f(x)$.

Theorem 1 follows from Theorem 4 and the following:

LEMMA. *Let* **F** *be the family of all functions of class* α *on a Polish space X. Then* **F** *is a complete ordinary function system and the sets of the form* $\{x\colon f(x) > c\}$, $f \in \mathbf{F}$, *c real, are just the sets of additive Borel class* α.

PROOF. It is shown in [3] that **F** forms a complete ordinary function system.

Any set of the form $\{x\colon f(x) > c\}$, $f \in \mathbf{F}$, c real, is clearly of additive Borel class α. Let A be any set of additive Borel class α. If $\alpha = 0$, A is a cozero set and hence $A = \{x\colon f(x) > 0\}$ for some continuous function f. Let $\alpha > 0$, then we can write $A = \bigcup_{n=1}^{\infty} A_n$ where the A_n's are ambiguous of class α. Let $f(x) = \sum_{n=1}^{\infty} 2^{-n} I_{A_n}(x)$ where I_{A_n} denotes the indicator function of A_n. As I_{A_n} is of class α, f is of class α and $A = \{x\colon f(x) > 0\}$.

3. **Proof of Theorem 2.** If $f(x) = \sup_y g(x,y)$ where g is Borel measurable, it is shown in [3] that f is an A-function. For this, f need not be bounded below.

Let f be an A-function on X such that $f(x) > a$ for a fixed real number a. Without loss of generality, we take $X = R$. Let $\{r_n\}$ enumerate all rationals. Let $A = \{(x,y)\colon f(x) > y\}$. Then $A = \bigcup_n\{(x,y)\colon f(x) > r_n > y\}$ and hence is analytic. Let $B \subset R^3$ be a Borel set such that $A =$ projection of B i.e. $(x,y) \in A \Leftrightarrow \exists z((x,y,z) \in B)$. Let $k\colon R^3 \to R^3$ be defined by

$$k(x,y,z) = \begin{cases} (x,y,z) & \text{if } (x,y,z) \in B, \\ (a,a,a) & \text{otherwise} . \end{cases}$$

Then, as k is Borel measurable so is $\pi_2 k$ where π_2 denotes projection to the second coordinate and

$$\pi_2 k(x,y,z) = \begin{cases} y & \text{if } (x,y,z) \in B, \\ a & \text{otherwise}. \end{cases}$$

Thus $\sup_{(y,z)} \pi_2 k(x,y,z) = \sup_{(y,z)}\{\{y: y < f(x)\} \cup \{a\}\} = f(x)$. Let ϕ be a Borel isomorphism from R onto R^2. Let $h: R^2 \to R^3$ be defined by $h(x,y) = (x, \phi(y))$ and let $g(x,y) = \pi_2 kh(x,y)$. Then g is Borel measurable and $f(x) = \sup_y \pi_2 k(x, \phi(y)) = \sup_y g(x,y)$.

REMARK. It is easy to see that Theorem 2 holds even if the condition "f is bounded below" is replaced by "f dominates a Borel function". Thus an A-function is of the form $\sup_y g(x,y)$ for some Borel measurable g if, and only if, it dominates a Borel function. Equivalently, every A-function is of the form $\sup_y g(x,y)$ for some Borel measurable g if, and only if, given an ascending sequence of analytic sets $\{A_n\}$ such that $\bigcup_{n=1}^{\infty} A_n = X$, there is an ascending sequence $\{B_n\}$ of Borel sets such that $B_n \subset A_n$ and $\bigcup_{n=1}^{\infty} B_n = X$. However, we do not know if this condition always holds.

4. **Proof of Theorem** 3. In X, we put $\mathbf{S}_0$ = the family of open sets, $\mathbf{B}_0 = \sigma(\mathbf{S}_0)$ and, for $0 < \alpha < \omega_1$, $\mathbf{S}_\alpha = \mathcal{A}(\sigma(\bigcup_{i<\alpha} \mathbf{S}_i))$ and $\mathbf{B}_\alpha = \sigma(\mathbf{S}_\alpha)$ where, for any family of sets $\mathbf{G}$, $\sigma(\mathbf{G})$ denotes the σ-algebra generated by $\mathbf{G}$ and $\mathcal{A}(\mathbf{G})$ denotes the smallest family containing $\mathbf{G}$ and closed under operation A. We call $(\mathbf{S}_\alpha, *)$ functions S_α-functions. Theorem 3 is obtained from the following more general theorem by putting $\alpha = 1$.

THEOREM 5. *On any uncountable Polish space X, there is an S_α-function f and there is an analytic subset C of the real line such that $f^{-1}(C) \notin \mathbf{B}_\alpha$.*

PROOF. It is known that $\mathbf{B}_\alpha$ is not closed under operation A (cf. [2]). Let $\{Z_{n_1 \cdots n_k}\} \subset \mathbf{B}_\alpha$ be such that $\bigcup_{n \in \mathfrak{N}} \bigcap_{k=1}^{\infty} Z_{n_1 \cdots n_k} \notin \mathbf{B}_\alpha$, where $\mathfrak{N}$ denotes the family of all sequences of positive integers and $n = (n_1, n_2, \ldots)$. We can find countably many sets $\{A_i\}$ in $\mathbf{S}_\alpha$ such that for all n and k, $Z_{n_1 \cdots n_k} \in \sigma(\{A_i\})$. Let $f(x) = \sum_{i=1}^{\infty} (2/3^i) I_{A_i}(x)$. As the sum of two S_α-functions, a positive constant multiple of an S_α-function and the limit of an increasing sequence of S_α-functions are all S_α-functions, f is an S_α-function. As $f^{-1}(\mathbf{B}) = \sigma(\{A_i\})$ where $\mathbf{B}$ is the Borel σ-algebra on R, we can find, for all n and k, $B_{n_1 \cdots n_k} \in \mathbf{B}$ such that $f^{-1}(B_{n_1 \cdots n_k}) = Z_{n_1 \cdots n_k}$. Let $C = \bigcup_{n \in \mathfrak{N}} \bigcap_{k=1}^{\infty} B_{n_1 \cdots n_k}$. Then C is analytic and $f^{-1}(C) = \bigcup_{n \in \mathfrak{N}} \bigcap_{k=1}^{\infty} Z_{n_1 \cdots n_k} \notin \mathbf{B}_\alpha$.

REMARK. Let X be any set and $\mathbf{L}$ a σ-additive lattice on X containing X and the null set, such that $\sigma(\mathbf{L})$ is not closed under operation A. We call a real valued function f on X an $\mathbf{L}^*$-function if for every c, $\{x: f(x) > c\} \in \mathbf{L}$. Evidently $f^{-1}(\mathbf{B}) \subset \sigma(\mathbf{L})$. However, we can find an analytic set C and an $\mathbf{L}^*$-function f such that $f^{-1}(C) \notin \sigma(\mathbf{L})$. The proof is similar to that of Theorem 5.

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Statistics-Mathematics Division, Indian Statistical Institute, Calcutta, India