COMPACT COMPOSITION OPERATORS ON $B(D)$

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Abstract. Let $D$ be a domain in the complex plane, $\phi: D \to D$ be analytic, and $B(D)$ be the uniform algebra of bounded analytic functions on $D$ with maximal ideal space $M$. The composition operator $C_\phi(f) = f \circ \phi$ is compact if and only if the weak* and norm closures of $\phi(D)$ coincide if and only if whenever the Euclidean closure of $\phi(D)$ contains a point $\lambda$ of the boundary of $D$ then each $f \in B(D)$ extends continuously from $\phi(D)$ to $\lambda$. If $C_\phi$ is compact, then either $\phi$ fixes a point of $D$ or else the adjoint of $C_\phi$ fixes a point of $M$.

Introduction. Let $D$ be a domain in the complex plane which supports nonconstant bounded analytic functions and let $B(D)$ be the uniform algebra of bounded analytic functions on $D$ with supremum norm. Each analytic $\phi: D \to D$ defines the composition operator $C_\phi$ on $B(D)$ by $C_\phi(f) = f \circ \phi$ for all $f \in B(D)$. Each composition operator is clearly linear and norm reducing.

This paper consists of two parts. In §1 we characterize compact composition operators on $B(D)$, and in §2 we discuss fixed points of $\phi$ when $C_\phi$ is compact.

1. Compact operators. For each $z \in D$ denote by $z$ the evaluation functional on $B(D)$ defined by $z(f) = f(z)$ for each $f \in B(D)$. We can then consider $D$ as a subset of $B(D)^*$. For each $C_\phi$ denote by $\Phi: B(D)^* \to B(D)^*$ the adjoint of $C_\phi$ defined by

$$\Phi(T)(f) = T(C_\phi(f)), \quad f \in B(D), \quad T \in B(D)^*,$$

so that if $T$ is $z$ for any $z \in D$ we have

$$\Phi(z)(f) = \hat{z}(C_\phi(f)) = \hat{z}(f \circ \phi) = f(\phi(z)) = (\phi(z))^* f,$$

and the function $\phi$ is the restriction of $\Phi$ to $D$.

We assume that each point $\lambda$ in the boundary $\partial D$ of $D$ is essential for $B(D)$ in that there is some $f \in B(D)$ which does not extend to be analytic at $\lambda$. The domain $D$ comes equipped with the usual topology from the plane induced by the chordal metric so that every closed subset of $D$ is compact. $D$ also inherits

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1 The material in this paper formed part of the author's dissertation prepared under Stephen D. Fisher.
both the weak* and norm topologies from $B(D)^*$, and all three topologies agree inside $D$. For each subset $A$ of $D$ denote by $\text{cl } A$ and $w^* - \text{cl } A$ the Euclidean and weak* closures of $A$.

$D$ is a subset of the maximal ideal space $M$ of $B(D)$ but does not exhaust all of $M$. For each $\lambda \in \text{cl } D$ the fiber $M_\lambda$ over $\lambda$ is the set of all $m \in M$ for which $m(f) = f(\lambda)$ whenever $f \in B(D)$ extends to be analytic at $\lambda$. Denote by $B_1$ the closed unit ball of $B(D)$. A linear operator $L$ on $B(D)$ is called compact if $L(B_1)$ is relatively norm compact. Finally if $\{f_n\}$ is a sequence in $B(D)$ with $f_n \to f \in B(D)$ uniformly on compact subsets of $D$ we write $f_n \to f$ ucc.

The following theorem of H. J. Schwartz [4, Theorem 2.5] can be proved by a simple normal families argument.

1.1. THEOREM. A composition operator $C_\phi$ is compact on $B(D)$ if and only if for every sequence $\{f_n\}$ in $B_1$ with $f_n \to 0$ ucc we have $\|f_n \circ \phi\| \to 0$.

Call a set $A$ in $M$ a peak set for $B(D)$ if there is some $f \in B_1$ whose Gel'fand transform $\hat{f}$ is equal to 1 on $A$ while $|\hat{f}(m)| < 1$ for all $m \in M - A$.

1.2. COROLLARY. Let $D$ be a domain for which the fiber $M_\lambda$ is a peak set for $B(D)$ for every $\lambda \in \partial D$. Then $C_\phi$ is compact on $B(D)$ if and only if $\text{cl } \phi(D)$ contains no point of $\partial D$.

When there is a $\lambda \in \partial D$ whose fiber is a nonpeak set the situation is more complicated. T. W. Gamelin and J. Garnett [2] showed that if $M_\lambda$ is not a peak set, then there is an unique $m_\lambda \in M_\lambda$ called the distinguished homomorphism with a representing measure living in $M - M_\lambda$. If we denote by $P(m_\lambda, \varepsilon)$ the open $\varepsilon$-ball about $m_\lambda$ in the norm of $B(D)^*$, then $P(m_\lambda, \varepsilon) \cap D$ is nonempty for all $\varepsilon > 0$. Moreover $\{\lambda\}$ is a singleton component of $\partial D$.

We call a sequence $\{z^n\}$ in $D$ an interpolating sequence if for every $\{s^n\} \in l^\infty$ there is some $f \in B(D)$ with $f(z^n) = s_n$ for all $n$. Interpolating sequences and distinguished homomorphisms are related by the following theorem [2, Theorem 3.5].

1.3. THEOREM. If $\{z^n\}$ is a sequence in $D$ which converges to some $\lambda \in \partial D$, then either $\{z^n\}$ contains an interpolating sequence or else $M_\lambda$ is a nonpeak set, and $\{z^n\}$ converges to the distinguished homomorphism $m_\lambda$ in the norm of $B(D)^*$.

1.4. COROLLARY. The closure of $D$ in the norm of $B(D)^*$ is the union of $D$ and the set of distinguished homomorphisms.

1.5. COROLLARY. If $C_\phi$ is compact on $B(D)$, then $\phi(D)$ contains no interpolating sequences.

Denote by $\Lambda$ the set of distinguished homomorphisms, and for each $\varepsilon > 0$ define

$$K_\varepsilon = w^* - \text{cl } D - \bigcup_{m_\lambda \in \Lambda} P(m_\lambda, \varepsilon).$$

1.6. THEOREM. The following are equivalent:

(a) $C_\phi$ is compact on $B(D)$.

(b) The norm and weak* closures of $\phi(D)$ coincide.

(c) The only weak* cluster points of $\phi(D)$ in $M - D$ are distinguished homomorphisms.
(d) For every $\epsilon > 0$ the set $K_\epsilon$ is a compact subset of $D$.

**Proof.** (a) implies (b). Let $C_\phi$ be compact. We show that every weak* cluster point of $\phi(D)$ is a norm cluster point. If $m \in M$ is a weak* cluster point of $\phi(D)$ there is a net $(\phi(z_\alpha))$ converging weak* to $m$. The net $(z_\alpha)$ is an infinite subset of the weak* compact $M$ and therefore has a subnet $(z_\beta)$ converging weak* to some $m_\star \in M$. The net $(z_\beta)$ is bounded and weak* convergent and $C_\phi$ is compact, so by [1, Theorem 6, p. 486], $(\Phi(z_\beta)) = ((\phi(z_\beta))^*)$ converges in norm $\Phi(m_\star)$. At the same time $(\phi(z_\beta))$ converges weak* to $m$ so $\Phi(m_\star) = m$.

(b) implies (c) implies (d). Trivial.

(d) implies (a). Let $\epsilon > 0$. Without loss of generality we can assume $\epsilon$ so small that $K_{\epsilon/8}$ is nonempty. Let $(f_n)$ be a sequence in $B_1$ with $f_n \to 0$ ucc. $K_{\epsilon/8}$ is a nonempty compact subset of $D$, so there exists a natural number $N$ such that $n > N$ implies $|f_n(z)| < \epsilon/2$ for all $z \in K_{\epsilon/8}$. Each $z \in \phi(D) - K_{\epsilon/8}$ lies in some $P(m_\lambda, \epsilon/8)$, and since $\phi(D)$ is connected the sets $K_{\epsilon/8}$ and $P(m_\lambda, \epsilon/4)$ must overlap. For any $z \in K_{\epsilon/8} \cap P(m_\lambda, \epsilon/4)$ and $w \in P(m_\lambda, \epsilon/8)$ we have at the same time $|f_n(z)| < \epsilon/2$ and $|f_n(z) - f_n(w)| < \epsilon/2$ whenever $n \geq N$, so that $|f_n(w)| < \epsilon$. The union of $K_{\epsilon/8}$ and all the sets $P(m_\lambda, \epsilon/8)$ covers $\phi(D)$, so we have $|f_n(z)| < \epsilon$ for any $z \in \phi(D)$ whenever $n \geq N$, and therefore $C_\phi$ is compact by Theorem 1.1.

1.7. **Theorem.** $C_\phi$ is compact on $B(D)$ if and only if whenever the Euclidean closure of $\phi(D)$ contains a point $\lambda \in \partial D$ then $\lambda$ possesses the distinguished homomorphism $m_\lambda$ and each $f \in B(D)$ extends weak* continuously from $\phi(D)$ to $\lambda$ according to $f(\lambda) = m_\lambda(f)$.

**Proof.** If $C_\phi$ is compact on $B(D)$ and $\overline{\phi(D)}$ contains $\lambda \in \partial D$, then $M_\lambda$ must be a nonpeak set with distinguished homomorphism $m_\lambda$. Let $(z_n)$ be a sequence in $\phi(D)$ with $z_n \to \lambda$. Corollary 1.5 says that $(z_n)$ cannot contain an interpolating sequence, so by Theorem 1.3 $(z_n)$ converges in norm to $m_\lambda$. By part (b) of Theorem 1.6 the weak* closure of $\phi(D)$ contains no other points of $M_\lambda$ and each $f \in B(D)$ extends weak* continuously from $\phi(D)$ to its weak* closure and therefore from $\phi(D)$ on $\lambda$ according to $f(\lambda) = m_\lambda(f)$.

Conversely suppose $C_\phi$ is not compact. Then by part (c) of Theorem 1.6, $\phi(D)$ must have a weak* cluster point $m \in M - D$ which is not a distinguished homomorphism. Then $m \in M_\lambda$ for some $\lambda \in \partial D$. If $\lambda$ does not possess a distinguished homomorphism, we are done. If there is $m_\lambda \in M_\lambda$ then $m \neq m_\lambda$.

If on the one hand $m_\lambda$ is also a weak* cluster point of $\phi(D)$ there are nets $(z_\alpha)$ and $(w_\beta)$ in $D$ with $(\phi(z_\alpha))$ and $(\phi(w_\beta))$ converging to $m$ and $m_\lambda$ respectively. Choose any $f \in B(D)$ with $f(m) \neq f(m_\lambda)$. Then this $f$ has distinct weak* limits at $\lambda$.

If on the other hand $m_\lambda$ is not a weak* cluster point of $\phi(D)$ then any $(\phi(z_n))$ converging to $\lambda$ contains an interpolating sequence $(\phi(z_n))$ by Theorem 1.3, so there is an $f \in B(D)$ with $f(\phi(z_n)) = (-1)^k$, and this $f$ is not continuous at $\lambda$.

We can now construct an example of a compact composition operator $C_\phi$ for which $\overline{\phi(D)}$ contains a point of $\partial D$.

1.8. **Example.** The earliest examples of domains with nonpeak fibers are the $L$-domains studied by L. Zalcman [5]. An $L$-domain is a domain obtained by
excising from the punctured unit disc a sequence of disjoint closed discs \( \Delta(x_n, r_n) \) whose centers \( \{x_n\} \) are contained in the positive x-axis and accumulate only at 0. Zalcman showed that if \( \sum r_n/x_n < \infty \), then \( M_0 \), the fiber over 0, is a nonpeak set, and the complex measure \( \mu \) defined on \( \partial D \) by \( d\mu = \xi^{-1}d\xi \) is finite and defines the distinguished homomorphism \( m_0 \) by

\[
m_0(f) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(\xi)}{\xi} d\xi.
\]

Then \( m_0 \) has a representing measure with no mass in \( M_0 \), and if \( \Delta \) is a wedge in \( D \) centered on the negative x-axis with a vertex at 0, \( \phi \) the restriction to \( D \) of a Riemann map from the disc to an open wedge in \( D \) with a vertex at 0. Then \( 0 \in \text{cl} \phi(D) \), but \( \Phi_\phi \) is compact by Theorem 1.6 since each \( K_\varepsilon \) is a compact subset of \( D \).

2. Fixed points. For each analytic \( \phi: D \to D \) we define the iterates \( \phi_n \) of \( \phi \) by \( \phi_0(z) = z \), \( \phi_{n+1}(z) = \phi(\phi_n(z)) \). If there is a point \( w \in D \) such that \( \phi(w) = w \) and \( \phi_n(z) \to w \) for all \( z \in D \) we call \( w \) an attractive fixed point of \( \phi \).

2.1. Theorem. If \( \Phi_\phi \) is compact on \( B(D) \) then either \( \phi \) has an attractive fixed point in \( D \) or else there is an unique \( \lambda \in \partial D \) with distinguished homomorphism \( m_\lambda \) such that \( \phi_n(z) \to \lambda \) for all \( z \in D \), and \( \Phi(m_\lambda) = m_\lambda \).

Proof. Suppose \( \Phi_\phi \) is compact and \( \phi \) has no fixed point in \( D \). We know \( \phi \) cannot be a conformal automorphism of \( D \), so according to a theorem of M. H. Heins [3, Theorem 2.2] there is a set \( A \) in \( \partial D \) with \( \{\phi_n(z)\} \) converging to \( A \) in the sense that all the limit points of \( \{\phi_n(z)\} \) are contained in \( A \) for all \( z \in D \). \( A \) is either a singleton or a continuum. Since by Corollary 1.5 \( \phi(D) \) contains no interpolating sequences \( A \) must contain only points with nonpeak fibers by Theorem 1.4. Each such point is a singleton component of \( \partial D \), so there must be an unique \( \lambda \in \partial D \) with distinguished homomorphism \( m_\lambda \) such that \( \phi_n(z) \to \lambda \) for every \( z \in D \). Furthermore \( \{(\phi_n(z))\} \) converges to \( m_\lambda \) in norm.

Now \( \Phi \) is norm continuous so by Corollary 1.4 \( \Phi(m_\lambda) \) must be either a point of \( D \) or a distinguished homomorphism. If \( \Phi(m_\lambda) \) is a distinguished homomorphism it must be \( M_\lambda \) itself, and we are done.

If \( \Phi(m_\lambda) = z_0 \in D \) we define the iterates of \( \Phi \) in the same way we defined the iterates of \( \phi \), so that for \( z \in D \) we have \( \Phi_n(z) = (\phi_n(z)) \) \( \Phi_\lambda \). Then \( \Phi_{n+1}(m_\lambda) = (\phi_n(z_0)) \) \( \Phi_\lambda \), and \( \{(\phi_n(z))\} \) converges in norm to \( m_\lambda \), but

\[
\Phi_{n+1}(m_\lambda) = \Phi(\Phi_n(m_\lambda)) \to \Phi(m_\lambda)
\]

in norm also, and we must have \( \Phi(m_\lambda) = m_\lambda \) contradicting \( \Phi(m_\lambda) = z_0 \).

2.2. Example. We show that there are functions \( \phi \) without fixed points whose composition operators are compact. Let \( \frac{1}{3} < r < 1 \) and \( \phi(z) = rz \). We construct an \( L \)-domain \( D \) so that \( \phi(D) \subset D \) and \( C_\phi \) is compact on \( B(D) \), but \( \phi \) fixes no point of \( D \).

About \( r \) there is a closed disc \( \Delta_1 = \Delta(r, \varepsilon_1) \) such that \( \phi(\Delta_1) \) does not meet \( \Delta_1 \). Inside \( \phi(\Delta_1) \) there is a disc \( \Delta(r^2, \varepsilon_2) \). Let \( \Delta_2 = \Delta(r^2, \varepsilon_2/16) \). Then inside \( \phi(\Delta_2) \)
there is another disc $\Delta(r^3, \epsilon_3)$. Let $\Delta_3 = \Delta(r^3, \epsilon_3/64)$, and so on with $\Delta_n = \Delta(r^n, 2^{-2n}\epsilon_n)$. Let $D$ be the complement in the punctured disc of the union of the $\Delta_n$'s. Then $\phi(D) \subset D$, and $\text{cl} \phi(D)$ contains $0 \in \partial D$. Each $\epsilon_n < 1$, and $\frac{1}{2} < r < 1$, so

$$\sum_{n=1}^{\infty} \frac{2^{-2n}\epsilon_n}{r^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^{2n}r^n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty$$

and $M_0$ is a nonpeak set by [5, p. 255].

Then the Cauchy integral formula [5, §4] produces a series expansion for $f$ which can be shown to converge uniformly in $\phi(D) \cup \{0\}$ by imitating the proof of [5, Theorem 5.2], so that $C_\phi$ is compact by Theorem 1.7.

**References**


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