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A REMARK ON THE RESTRICTION MAP IN FIELD FORMATION

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ABSTRACT. In this note we point out that in a field formation $(G, \{G_F\}, A)$, if $h_2(K/F) = [K: F]^c$ for every normal layer K/F with a fixed integer $c \geq 0$, then for every tower $F \subset E \subset K$ with K/F normal, the restriction map $H^2(K/F) \to H^2(K/E)$ is surjective, and give an example with $c = 2$.

Let $(G, \{G_F\}, A)$ be a field formation. (For the notations and basic facts, see [1, Chapter 14, pp. 197-209].) Then for every tower $F \subset K \subset L$ with K/F and L/F normal, the sequence

$$(1) \qquad 0 \to H^2(K/F) \xrightarrow{\text{inf}} H^2(L/F) \xrightarrow{\text{res}} H^2(L/K)$$

is exact. We shall prove that if $h_2(K/F) = [K: F]^c$ for every normal layer K/F with a fixed integer $c \geq 0$, then for every tower $F \subset E \subset K$ with K/F normal, res: $H^2(K/F) \to H^2(K/E)$ is surjective.

If $F \subset K \subset L$ with K/F and L/F normal, then the exact sequence (1) gives that the order of the image res $H^2(L/F)$ is equal to $[L: F]^c/[K: F]^c = [L: K]^c = h_2(L/K)$. Thus res: $H^2(L/F) \to H^2(L/K)$ is surjective in this case.

Let $F \subset E \subset K$ with K/F normal. For each prime p, the restriction map takes the p-primary component $H^2(K/F)_p$ of $H^2(K/F)$ into the p-primary component $H^2(K/E)_p$ of $H^2(K/E)$. Thus it is enough to show that res: $H^2(K/F)_p \to H^2(K/E)_p$ is surjective for every p.

Let G_{K/E_0} be a p-Sylow subgroup of $G_{K/E}$. Then by a Sylow theorem, there exists a chain

$$G_{K/E_0} = G_{K/F_r} \subset G_{K/F_{r-1}} \subset \cdots \subset G_{K/F_0}$$

of p-subgroups of $G_{K/F}$ such that G_{K/F_i} is normal in $G_{K/F_{i-1}}$ for each $i = 1, \ldots, r$ and G_{K/F_0} is a p-Sylow subgroup of $G_{K/F}$. We know that the restriction map takes $H^2(K/E)_p$ injectively into $H^2(K/E_0)$. In our case,

$$\text{res: } H^2(K/E)_p \to H^2(K/E_0)$$

is bijective because both $H^2(K/E)_p$ and $H^2(K/E_0)$ have the same order $[K: E_0]^c$. Likewise

$$\text{res: } H^2(K/F)_p \to H^2(K/F_0)$$

is bijective. Thus it is sufficient to show that

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(2) $$\text{res: } H^2(K/F_0) \to H^2(K/E_0)$$

is surjective. This map factors as

$$H^2(K/F_0) \xrightarrow{\text{res}} H^2(K/F_1) \xrightarrow{\text{res}} \cdots \xrightarrow{\text{res}} H^2(K/F_r).$$

Since K/F_i and F_i/F_{i-1} are normal, each factor

$$\text{res: } H^2(K/F_{i-1}) \to H^2(K/F_i)$$

is surjective. Thus the composite (2) of these is surjective. This completes the proof of the remark.

We now offer an example of field formation in which $h_2(K/F) = [K: F]^2$ for every normal layer K/F. Let p be a rational prime and $\mathbf{Q}_p$ be the rational p-adic number field. Let $P = \mathbf{Q}_p\{t\}$, the field of formal power series in t over $\mathbf{Q}_p$. Let Ω be the splitting field of the polynomials $X^n - t$ over P for all integers $n > 0$ not divisible by p. Given a finite extension F of P in Ω, let G_F be the Galois group of Ω/F. Then $(G, \{G_F\}, \Omega^\times)$, where $G = G_P$ and $\Omega^\times$ is the multiplicative group of Ω, is a field formation. We claim that $h_2(K/F) = [K: F]^2$ for every normal layer K/F in Ω/P.

The ground field P is complete under the nonarchimedean valuation $|\ |$ given by $|x| = e^{-r}$ if

$$x = \sum_{k \geq r} a_k t^k, \qquad a_k \in \mathbf{Q}_p,\ a_r \neq 0,$$

and the valuation is extended to Ω. Given a field K in the formation, let $\mathcal{O}_K$, M_K and U_K be the valuation ring, its maximal ideal and the group of units in $\mathcal{O}_K$, respectively. Let $\overline{K} = \mathcal{O}_K/M_K$ and $U_K^1 = 1 + M_K$. The residue field $\overline{K}$ is an abelian extension of $\mathbf{Q}_p$.

Since every normal layer in our formation is solvable, by induction we see that it is sufficient to prove the equality $h_2(K/F) = [K: F]^2$ for every cyclic layer K/F of prime degree. For this it is sufficient to establish the equality in the following two cases: (a) when K/F is unramified and (b) when K/F is totally ramified.

For any normal layer K/F, we have $H^q(G_{K/F}, U_K^1) = 0$ for all q because $x \mapsto x^n$ is an automorphism of U_K^1. Thus the exact sequence

$$0 \to U_K^1 \to U_K \to \overline{K}^\times \to 0$$

gives that

(3) $$H^q(G_{K/F}, U_K) = H^q(G_{K/F}, \overline{K}^\times)$$

for all q. While the exact sequence

(4) $$0 \to U_K \to K^\times \xrightarrow{\nu_K} \mathbf{Z} \to 0,$$

where ν_K is the exponential valuation on K, gives the long exact sequence

(5) $$0 \to H^2(G_{K/F}, U_K) \to H^2(K/F) \to H^2(G_{K/F}, \mathbf{Z}) \xrightarrow{\delta} \cdots.$$

Note that

$$H^2(G_{K/F}, \mathbf{Z}) = H^1(G_{K/F}, \mathbf{Q}/\mathbf{Z}) = G^*_{K/F},$$

the character group of $G_{K/F}$.

CASE (a). K/F is unramified. Since $G_{K/F} = G_{\overline{K}/\overline{F}}$, (3) gives that

$$H^q(G_{K/F}, U_K) = H^q(\overline{K}/\overline{F}).$$

Since the exact sequence (4) splits in this case, we get the exact sequence

$$0 \to H^2(\overline{K}/\overline{F}) \to H^2(K/F) \to G^*_{K/F} \to 0.$$

Since K/F is abelian, $|G^*_{K/F}| = [K: F]$. While by the local class field theory, $h_2(\overline{K}/\overline{F}) = [\overline{K}: \overline{F}] = [K: F]$. Thus we get that $h_2(K/F) = [K: F]^2$.

CASE (b). K/F is totally ramified. Then $\overline{K} = \overline{F}$ and this field contains a primitive nth root of unity, where $n = [K: F]$, and K/F is cyclic. Thus

$$H^3(G_{K/F}, \overline{K}^\times) = H^1(G_{K/F}, \overline{F}^\times)$$

is of order n. Since $H^3(K/F) = H^1(K/F) = 0$ and $|G^*_{K/F}| = n$,

$$\delta\colon G^*_{K/F} \to H^3(G_{K/F}, \overline{K})$$

is an isomorphism. Thus (5) gives that

$$H^2(K/F) = H^2(G_{K/F}, \overline{K}^\times) = H^0(G_{K/F}, \overline{F}^\times) = \overline{F}^\times/\overline{F}^{\times n}.$$

But since $(n, p) = 1$, $[\overline{F}^\times: \overline{F}^{\times n}] = n^2$. Thus $h_2(K/F) = [K: F]^2$.

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