WHEN IS $D + M$ COHERENT?

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Abstract. Let $V$ be a valuation ring of the form $K + M$, where $K$ is a field and $M (\neq 0)$ is the maximal ideal of $V$. Let $D$ be a proper subring of $K$. Necessary and sufficient conditions are given that the ring $D + M$ be coherent. The condition that a given ideal of $V$ be $D + M$-flat is also characterized.

1. Introduction and notation. Let $V$ be a valuation ring of the form $K + M$, where $K$ is a field and $M (\neq 0)$ is the maximal ideal of $V$. Let $D$ be a proper subring of $K$; let $k$, viewed inside $K$, be the quotient field of $D$. Our purpose is twofold: to answer the question raised in the title, and to determine when a given ideal of $V$ is a flat $D + M$-module. Entering into the solution is a result of Ferrand [6] on the descent of flatness.

It is well known (cf. [7, Theorem A(m), p. 562]) that $D + M$ is Noetherian (and, hence, coherent) if and only if the following hold: $D = k$, $K$ is a finite (algebraic) extension of $k$, and $V$ is discrete rank one. One upshot of considering coherence for $D + M$ is the relaxing of the first and third of these conditions. (See Theorem 3 and Remark 6(a) below.) To motivate the second problem, note that the riding homological assumption used by Greenberg and Vasconcelos [10] to study coherence for a certain family of pullbacks is, when specialized to the $D + M$-construction, the condition that ($k = K$ and) $M$ is $D + M$-flat. Inasmuch as $V$ is coherent and $M$ is $V$-flat, it is striking (see Corollary 8) that, in case $D = k$, the coherence of $D + M$ forces $M$ to be nonflat over $D + M$.

Background material on the $D + M$-construction and coherence may be found in [7, Appendix 2] and [2], respectively. In addition to the notation fixed above, it will be convenient to denote $D + M$ by $R$.

2. Coherence and flatness. Before answering the question raised in the title, we give two lemmas.

Lemma 1. Let $M$ be a finitely generated ideal of $R$. Then $D = k$.

Proof. Since $M \neq 0$, it follows from Nakayama's lemma that $M \neq M^2$. Now $M$ is cyclic as a $V$-module (since $V$ is valuation), so that $M/M^2$ is cyclic over $V/M = K$. Thus $M/M^2$ and $K$ are isomorphic as $K$-spaces and, a fortiori,
as $D$-modules. However, $M/M^2$ is finitely generated over $R/M = D$, so that $K$ is a finitely generated $D$-module. By integrality [1, Lemma 2, p. 326], $D$ is a field, as required.

An integral domain $T$, with quotient field $L$, is said to be finite-conductor in case $Ta \cap Tb$ is a finitely generated $T$-module for each $a, b$ in $L$. (Finite-conductor domains have figured recently in [11], [3], and [4].) By [2, Theorem 2.2], any coherent domain is finite-conductor.

**Lemma 2.** If $R$ is finite-conductor and $k \neq K$, then $M$ is a finitely generated ideal of $R$ and $D = k$.

**Proof.** Select $b$ in $K \setminus k$ and nonzero $m$ in $M$. We claim that $Rm \cap Rbm = Mm$. Indeed, containment one way is clear, as $M = Mb$. Conversely, if $r$ is in $Rm \cap Rbm$, then

$$r = (d_1 + m_1)m = (d_2 + m_2)b$$

for some $d_i$ in $D$, $m_i$ in $M$ ($i = 1, 2$). Cancellation gives $d_1 + m_1 = d_2b + m_2b$ and so $d_1 = d_2b$. Since $b$ is not in $k$, we have $d_2 = 0 = d_1$, so that $r = m_1m$ is in $Mm$, thus sustaining the claim. As $R$ is finite-conductor, $Mm$ is a finitely generated ideal of $R$. However, $Mm$ and $M$ are $\mathbb{Z}$-isomorphic, and an application of Lemma 1 completes the proof.

**Theorem 3.** $R$ is coherent if and only if one of the following conditions holds:

1. $k = K$ and $D$ is coherent;
2. $M$ is a finitely generated ideal of $R$.

Moreover, if condition (2) holds, then $D = k$ and $K$ is a finite extension of $k$.

**Proof.** Assume that $R$ is coherent. If $k = K$, an easy direct argument or an appeal to [8, Theorem 5.14] shows that $D$ is coherent. Now suppose that $k \neq K$. By Lemma 2, $M$ is finitely generated over $R$.

Conversely, we see directly or by [8, Theorem 5.14] that condition (1) implies that $R$ is coherent. Next, assume that (2) holds. By the proof of Lemma 1, $D = k$ and there is an integer $n > 2$ such that $K \cong M/M^2 \cong k^n$ isomorphic as $k$-spaces. (This yields the final assertion of the theorem.) In particular, $V$ is a finitely generated $R$-module.

We claim that $V$ is finitely presented over $R$. Let $\{b_i : 1 \leq i \leq n\}$ be any $k$-basis of $K$. If $R^n$ is $R$-free on a basis $\{e_i : 1 \leq i \leq n\}$, the $R$-module homomorphism $g : R^n \to V$ determined by $g(e_i) = b_i$ is surjective. It is straightforward to verify that the $R$-module ker$(g)$ is isomorphic to the direct sum of $n - 1$ copies of $M$. By (2), ker$(g)$ is finitely generated over $R$, thus establishing the above claim.

To show that $R$ is coherent, we use the criterion in [2, Theorem 2.1(a)]; viz., we show that the product of any family $\{A_j : j \in J\}$ of flat $R$-modules is flat. As each $A_j \otimes_R V$ is $V$-flat and $V$ is coherent, $\Pi(A_j \otimes_R V)$ is $V$-flat. However, since $V$ is finitely presented over $R$, the canonical homomorphism $(\Pi A_j) \otimes_R V \to \Pi(A_j \otimes_R V)$ is an isomorphism (cf. [1, Exercise 9(a), p. 43]), so that $(\Pi A_j) \otimes_R V$ is also $V$-flat. That $\Pi A_j$ is $R$-flat, now follows from Ferrand's descent result [6, Lemme], as applied to the inclusion $R \to V$, and completes the proof.
In view of [11, Theorem 1], the next result may be used to recover [7, Theorem A(i), p. 561].

**Corollary 4.** $R$ is integrally closed and coherent if and only if $k = K$ and $D$ is integrally closed and coherent.

**Proof.** Combine Theorem 3 with [7, Theorem A(b), p. 560] and [8, Theorem 5.14].

We next focus on condition (2) of Theorem 3, in order to prepare for the examples below.

**Corollary 5.** Let $D = k$. Then $R$ is coherent if and only if $K$ is a finite extension of $k$ and $M \neq M^2$.

**Proof.** By [9, Lemma 1.3], $M \neq M^2$ if and only if $M$ is a principal ideal of $V$. The “only if” half is now immediate from Theorem 3. Conversely, if $\{b_i : 1 \leq i \leq n\}$ is a (finite) $k$-basis of $K$ and $M = Vm$ for some $m$ in $M$, then $\{b_im : 1 \leq i \leq n\}$ is easily seen to generate $M$ as an $R$-module, and an application of Theorem 3 completes the proof.

**Remark 6.** (a) One may ask whether $k \neq K$ and $R$ coherent imply $R$ Noetherian. By Theorem 3 and Corollary 5 (and the result quoted in the introduction), the answer is affirmative if and only if $V$ has rank one.

To construct an instance where the answer is negative, let $K/k$ be a nontrivial finite field extension. As in [1, Example 6, p. 390], construct a valuation ring $V = K + M$ whose corresponding valuation $v$ has (rank two) value group $Z \times Z$, with the lexicographic order. As $M$ is the set of elements $b$ in the quotient field of $K + M$ such that $v(b) > 0$, every element $d$ in $M^2$ satisfies $v(d) \geq (0,2)$. Select $e$ with $v(e) = (0,1)$; then $e$ is in $M \backslash M^2$ and, by Corollary 5, $R = k + M$ is the desired example.

(b) The condition “$M \neq M^2$” in Corollary 5 cannot be deleted. Indeed, let $K/k$ again be nontrivial finite, with $V = K + M$ having value group $R$. Since $R = 2R$, it is clear that $M = M^2$, so that $R = k + M$ is not coherent. (To produce an example with $V$ of rank exceeding one, traffic similarly with the lexicographically ordered value group $R \times R$.)

**Theorem 7.** Let $I$ be a nonzero ideal of $V$. Then $I$ is $R$-flat if and only if at least one of the following conditions holds:

1. $k = K$;
2. $I$ is not a principal ideal of $V$.

**Note.** If condition (1) holds, then $I/MI$ is $D$-flat; by [9, Lemma 1.3],

(2) $\iff I = MI$.

**Proof.** Suppose that $k = K$. Then, $V = R_M$ is $R$-flat; moreover, $I$ is $V$-flat, since any ideal of $V$ is. Thus, transitivity of flatness shows that $I$ is $R$-flat, whence $I/MI \cong I \otimes_R D$ is $D$-flat.

Next, suppose that $I$ is $R$-flat and $k \neq K$. If (2) fails, then $I \neq MI$. Now $I \otimes_R (k + M) = I$ is a flat ideal of $k + M$, so that [12, Lemma 2.1] implies that $I$ is a principal ideal of $k + M$. Then $k \cong I/MI \cong K$, contradicting $k \neq K$. This concludes the “only if” part of the proof.

It remains to show that condition (2) guarantees that $I$ is $R$-flat. Let $a, b$ be elements of $I$; without loss of generality, $a$ divides $b$ in $V$. By (2), $I = MI$, so
that $a = \sum e_i d_i$, for some $e_i$ in $M$, $d_i$ in $I$. Without loss of generality, $e = e_1$ divides $e_i$ for each $i > 1$, so that $a$ and $b$ are each in $Ie \subseteq Re$. Thus, $I$ is the (filtered) direct limit of its principal subideals over $R$, hence is flat, to complete the proof.

Theorem 7 and Corollary 5 will be used by one of us in a subsequent paper in order to answer a question raised in [4] about rings of global dimension 3. Combining Theorem 3 with Theorem 7 (for the case $I = M$) leads immediately to our next result.

**Corollary 8.** $R$ is coherent and $M$ is $R$-flat if and only if $k = K$ and $D$ is coherent.

We close with a homological remark.

**Remark 9.** Let $R$ be coherent, such that $k \neq K$. Then $M$ has infinite projective dimension over $R$. For a proof, $D = k$ by Theorem 3, so that [5, Corollary] implies that $R$ is a going-down ring. If the result is denied, [4, Proposition 2.5] shows $R$ is Prüfer, and [7, Theorem A(i), p. 561] then yields $k = K$, the desired contradiction.

In view of Theorem 3 and Corollary 5, the next result generalizes the assertion in the preceding paragraph (and has a more straightforward proof). If $M \neq M^2$ and $k \neq K$, then $M$ has infinite flat (weak) dimension over $R$. For a proof, we may take $D = k$ since $M \otimes_R (k + M) = M$. By Theorem 7, $M$ is not $R$-flat, and so the proof of [4, Proposition 4.5] may be modified to give the desired result.

**References**