INTERPOLATION FOR ENTIRE FUNCTIONS OF EXPONENTIAL TYPE AND A RELATED TRIGONOMETRIC MOMENT PROBLEM

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ABSTRACT. A classical theorem of Hausdorff-Young shows that when $1 < p < 2$, the system of equations $\hat{\varphi}(n) = c_n (-\infty < n < \infty)$ admits a solution $\varphi$ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$. Here, as usual, $\hat{\varphi}$ denotes the complex Fourier transform of $\varphi$ and $q$ is the conjugate exponent given by $p^{-1} + q^{-1} = 1$. The purpose of this note is to show that if a set $\{\lambda_n\}$ of real or complex numbers is “sufficiently close” to the integers, then the corresponding system $\hat{\varphi}(\lambda_n) = c_n$ is also solvable for $\varphi$ whenever $\{c_n\} \in l^p$. The proof is accomplished by establishing a similar interpolation theorem for a related class of entire functions of exponential type.

1. Introduction. A classical theorem of Hausdorff-Young shows that when $1 < p < 2$, the system of equations $\hat{\varphi}(n) = c_n (-\infty < n < \infty)$ admits a solution $\varphi$ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$. Here, as usual, $\hat{\varphi}$ denotes the complex Fourier transform of $\varphi$ and $q$ is the conjugate exponent given by $p^{-1} + q^{-1} = 1$. In this note we show that if a set $\{\lambda_n\}$ of real or complex numbers is “sufficiently close” to the integers, then the corresponding system

\begin{equation}
\hat{\varphi}(\lambda_n) = c_n \quad (-\infty < n < \infty)
\end{equation}

admits a solution $\varphi$ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$. Specifically, we have the following result.

**Theorem 1.** Let $1 < p < 2$ and let $q$ be the conjugate exponent. There exists a constant $L > 0$ with the following property: If $|\lambda_n - n| \leq L$, then the system

\begin{equation}
\hat{\varphi}(\lambda_n) = c_n \quad (-\infty < n < \infty)
\end{equation}

admits a solution $\varphi$ in $L^q(-\pi, \pi)$ whenever $\{c_n\} \in l^p$.

We prove Theorem 1 by establishing a similar interpolation theorem for a related class of entire functions of exponential type.

2. Interpolation in a related Banach space of entire functions. We denote by $E^p_\tau (p \geq 1)$ the Banach space of entire functions of exponential type $\tau$ for which

$$
\|f\|_p = \left\{ \int_{-\infty}^{\infty} |f(x)|^p \, dx \right\}^{1/p} < \infty.
$$

A sequence $\{\lambda_n\}$ of distinct real or complex numbers is said to be an **interpolating sequence** for $E^p_\tau$ if $TE^p_\tau \supset l^p$, where $T$ is given by $Tf = \{f(\lambda_n)\}$. (Such sequences were studied extensively in [5] for the special cases $p = 1, 2$.

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and the limiting case \( p = \infty \). For general properties of the spaces \( E^p \) see \([2]\).

It is well known that \( E^p \) is closed under differentiation and that

\[
\|f\|_p \leq \tau\|f\|_p.
\]

A simple application of the closed graph theorem shows that if \( \{\lambda_n\} \) is an interpolating sequence, then the unit ball in \( L^p \) can be interpolated in a uniformly bounded way \([5]\); that is, there exists a constant \( M \) such that whenever \( c \in L^p, \|c\| \leq 1 \), there corresponds at least one function \( f \in E^p \) for which \( Tf = c \) and \( \|f\| \leq M \). As a consequence \([5]\), if the imaginary part of \( \lambda_n \) is uniformly bounded, then the \( \lambda_n \) are of necessity separated, that is \( \inf |\lambda_n - \lambda_m| > 0 \) \((n \neq m)\). But then, it follows that for every function \( f \) in \( E^p \),

\[
\left\{ \sum_n |f(\lambda_n)|^p \right\}^{1/p} \leq A\|f\|_p,
\]

where \( A = A(p, \tau, \{\lambda_n\}) \) is an absolute constant, independent of \( f \) \([6]\). Hence, in this case, \( T \) is in fact a bounded linear transformation into \( L^p \).

If \( S: E^p \to L^p \) is defined by \( Sf = \{ f(\mu_n) \} \), then we shall wish to conclude that \( SE^p = L^p \) knowing that \( TE^p = L^p \) and that \( S \) is "close" to \( T \). For this purpose, we will need the following interesting result of Bade and Curtis \([1]\).

**Lemma 1.** Let \( X \) and \( Y \) be Banach spaces and \( T: X \to Y \) a bounded linear transformation. Suppose that there exist constants \( M > 0 \) and \( 0 < \epsilon < 1 \) with the following property: For each \( y \) in the unit ball of \( Y \), there exists an \( x \) in \( X \) with \( \|x\| \leq M \) and \( \|Tx - y\| \leq \epsilon \). Then \( T \) is onto.

The proof of the following lemma is similar to that given in \([3]\) for the case \( p = 2 \).

**Lemma 2.** Let \( \{\lambda_n\} \) be a separated sequence of points lying in a strip parallel to the real axis, and suppose that \( |\mu_n - \lambda_n| \leq L \). Then for every function \( f \) belonging to \( E^p \), we have the inequality

\[
\left\{ \sum_n |f(\mu_n) - f(\lambda_n)|^p \right\}^{1/p} \leq A(e^{\epsilon L} - 1)\|f\|_p,
\]

where \( A \) is the same constant appearing in (3).

**Proof.** Using Taylor's theorem, we write

\[
f(\mu_n) - f(\lambda_n) = \sum_{k=1}^{\infty} \frac{f^{(k)}(\lambda_n)}{k!}(\mu_n - \lambda_n)^k.
\]

Then, for any \( \rho > 0 \),

\[
f(\mu_n) - f(\lambda_n) = \sum_{k=1}^{\infty} \frac{f^{(k)}(\lambda_n)}{\rho^k (k!)^{1/p}} \frac{\rho^k (\mu_n - \lambda_n)^k}{(k!)^{1/q}},
\]

and hence, by Hölder's inequality,

\[
|f(\mu_n) - f(\lambda_n)| \leq \left\{ \sum_{k=1}^{\infty} \frac{\|f^{(k)}(\lambda_n)\|_p}{\rho^k k!} \right\}^{1/p} \cdot \left\{ \sum_{k=1}^{\infty} \frac{\rho^k (\mu_n - \lambda_n)^k}{k!} \right\}^{1/q}.
\]
Now, since \( f^{(k)} \in E^p_T \), it follows from (2) and (3) that
\[
\sum_n |f^{(k)}(\lambda_n)|^p \leq A^p \tau^{kp} \|f\|_p^p.
\]

Therefore, we conclude that
\[
\left\{ \sum_n |f(\mu_n) - f(\lambda_n)|^p \right\}^{1/p} \leq A \|f\|_p \left\{ \sum_{k=1}^\infty \frac{\tau^{kp}}{\rho^k k!} \right\}^{1/p} \left\{ \sum_{k=1}^\infty \frac{(\rho L)^q}{k!} \right\}^{1/q}
\]
\[
= A \|f\|_p \{e^{\tau p} \rho^p - 1\}^{1/p} \{e^{\rho^q L^q} - 1\}^{1/q},
\]
and the result follows by taking \( \rho = \tau^{1/q} L^{-1/p} \).

The proof of Theorem 1 will follow easily from the following interpolation theorem for \( E^p_T \).

**Theorem 2.** Let \( \{\lambda_n\} \) be a sequence of points lying in a strip parallel to the real axis. If \( \{\lambda_n\} \) is an interpolating sequence for \( E^p_T \), then there exists a constant \( L > 0 \) such that \( \{\mu_n\} \) is also an interpolating sequence for \( E^p_T \) whenever \( |\mu_n - \lambda_n| < L \).

**Proof.** Since \( \{\lambda_n\} \) is interpolating for \( E^p_T \), the unit ball of \( l^p \) can be interpolated in a uniformly bounded way. Thus, there exists a constant \( M \) such that whenever \( \sum |c_n|^p < 1 \), there exists a function \( f \) in \( E^p_T \) with \( f(\lambda_n) = c_n \) and \( \|f\|_p \leq M \).

Let us define a mapping \( T : E^p_T \to l^p \) by \( Tf = \{f(\mu_n)\} \). The inequality (4) shows that \( T \) is a bounded linear transformation into \( l^p \). We show that \( T \) is in fact onto \( l^p \). Let \( c = \{c_n\} \) belong to the unit ball of \( l^p \) and choose \( f \) in \( E^p_T \) such that \( \|f\|_p \leq M \) and \( f(\lambda_n) = c_n \). Then (4) becomes
\[
\|Tf - c\| \leq AM(e^{\tau L} - 1),
\]
and since \( L \) can be chosen small enough so that the right-hand side of (5) is less than 1, the conclusion follows from Lemma 1.

**Corollary.** If \( 1 < p < 2 \), then \( \{\lambda_n\} \) is an interpolating sequence for \( E^p_T \) whenever \( |\lambda_n - \lambda| \leq L \) and \( L \) is sufficiently small.

**Proof.** In view of Theorem 2, it is enough to show that the integers are an interpolating sequence for \( E^p_T \). Suppose that \( \{c_n\} \in l^p \). By the Hausdorff-Young theorem, there exists a function \( \phi \) in \( L^q(-\pi, \pi) \) such that \( \hat{\phi}(n) = c_n \) \( (-\infty < n < \infty) \). Since \( \{\hat{\phi}(n)\} \in l^p \) and \( p > 1 \), it follows that \( \hat{\phi}(x) \in L^p(-\infty, \infty) \) [4]. Thus, the function \( \hat{\phi}(z) \) belongs to \( E^p_T \) and the proof is complete.

**Remark.** For \( p = 1 \), the integers fail to be an interpolating sequence for \( E^1_T \) for the trivial reason that the nonzero integers are already a set of uniqueness. It was shown in [5], however, that \( TE^1_T = l^1 \) for every \( \tau \neq \pi \).

3. **Proof of Theorem 1.** The proof of Theorem 1 follows immediately from the above corollary since every function \( f \) belonging to \( E^p_T \) is of the form \( f = \hat{\phi} \) for some \( \phi \) in \( L^q(-\pi, \pi) \) [2].
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References


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