WHICH OPERATORS ARE SIMILAR TO PARTIAL ISOMETRIES?

L. A. FIALKOW

Abstract. Let \( \mathcal{H} \) denote a separable, infinite dimensional complex Hilbert space and let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of all bounded linear operators on \( \mathcal{H} \). Let \( \mathcal{P} = \{ T \in \mathcal{L}(\mathcal{H}) | \sigma(T) < 1 \text{ and } T \text{ is similar to a partial isometry with infinite rank} \} \); let \( \mathcal{S} = \{ S \in \mathcal{L}(\mathcal{H}) | r(S) < 1, \text{range}(S) \text{ is closed, and } \text{rank}(S) = \text{nullity}(S) = n_0 \} \). It is conjectured that \( \mathcal{P} = \mathcal{S} \) and it is proved that \( \mathcal{P} \subset \mathcal{S} \subset \mathcal{P}^{-} \).

Introduction. Let \( \mathcal{H} \) denote a fixed separable, infinite-dimensional complex Hilbert space, and let \( \mathcal{L}(\mathcal{H}) \) denote the algebra of all bounded linear operators on \( \mathcal{H} \). In [5], Sz.-Nagy proved that an invertible operator \( T \) in \( \mathcal{L}(\mathcal{H}) \) is similar to a unitary operator if and only if the powers of \( T \) and \( T^{-1} \) are uniformly bounded; the proof of this result also implies that an operator is similar to an isometry if and only if its powers are uniformly bounded above and below [4]. In this note we state the following conjecture concerning operators similar to partial isometries, and then prove results which partially affirm the conjecture.

Conjecture. If \( T \) is an operator on \( \mathcal{H} \) with closed range, whose spectral radius is less than one, and such that \( \text{rank}(T) = \text{nullity}(T) = \text{nullity}(T^*) = n_0 \), then \( T \) is similar to a partial isometry.

Let \( \mathcal{P} = \{ T \in \mathcal{L}(\mathcal{H}) | r(T) < 1 \text{ and } T \text{ is similar to a partial isometry with infinite rank} \} \), where \( r(T) \) is the spectral radius of \( T \); let \( \mathcal{S} = \{ S \in \mathcal{L}(\mathcal{H}) | r(S) < 1, \text{range}(S) \text{ is closed, and } \text{rank}(S) = \text{nullity}(S) = \text{corank}(S) = n_0 \} \). It is easy to prove that \( \mathcal{P} \subset \mathcal{S} \) and in this note we prove that \( \mathcal{P} \subset \mathcal{S} \subset \mathcal{P}^{-} \) (the norm closure of \( \mathcal{P} \) in \( \mathcal{L}(\mathcal{H}) \)). To state the results in detail we use the following notation. If \( A \) and \( B \) are operators on \( \mathcal{H} \) such that \( A^*A + B^*B \) is invertible, let \( M(A, B) \) denote the operator on \( \mathcal{H} \oplus \mathcal{H} \) whose matrix is \( \begin{pmatrix} A & B \\ A^* & B^* \end{pmatrix} \); let \( \mathcal{T} \) denote the set of all matrices of this form whose spectral radii are less than one. Each operator in \( \mathcal{S} \) is unitarily equivalent to a matrix in \( \mathcal{T} \).

Theorem 1. The operator \( M(A, B) \) in \( \mathcal{T} \) is similar to a partial isometry if any of the following conditions are satisfied:

(i) \( 0 \) is not in the interior of \( \sigma(A) \);
(ii) \( \text{nullity}(A) = \text{corank}(A) \);
(iii) \( \text{nullity}(A) < \text{corank}(A) = n_0 \) and \( B \) is not compact;
(iv) \( B \) is a semi-Fredholm operator;
(v) \( \text{corank}(A) < \text{nullity}(A) \), \( A \) has closed range, and \( B^*|E \) is not compact, where

\[
E = \{ y \in \mathcal{H} | \exists x \in \mathcal{H} : A^*x + B^*y = 0 \}.
\]

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Let $(J)$ denote the ideal of all compact operators in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. If $T$ is in $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$, let $\tilde{T}$ denote the image of $T$ under the canonical homomorphism of $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})$ onto the Calkin algebra $\mathcal{L}(\mathcal{H} \oplus \mathcal{H})/(J)$.

**Theorem 2.** If $T = M(A, B)$ is in $\mathcal{S}$ then $\tilde{T}$ is similar to a partial isometry if either of the following conditions is satisfied:

(i) $\text{nullity}(A)$ and $\text{corank}(A)$ are finite;

(ii) $B$ is compact.

Because these results do not cover the case $\text{corank}(A) < \text{nullity}(A) = \aleph_0$, we do not know whether $\mathcal{S} = \mathcal{S}$. We note also that the proof of Theorem 1-iii was motivated by the proof of a factorization theorem of R. G. Douglas [1, Lemma 2.1]. The author thanks the referee for suggestions that have clarified certain points in the original proofs of our results.

**Proof of Theorems 1 and 2.**

**Lemma 0.** If $T = M(A, B)$ is in $\mathcal{S}$, then the nonzero elements of $\sigma(T)$ and $\sigma(B)$ are identical.

**Proof.** If $\lambda \neq 0$ and $B - \lambda$ is invertible, then a calculation shows that $(T - \lambda)^{-1}$ is given by the operator matrix

$$
\begin{pmatrix}
-1/\lambda & (-1/\lambda)A(B - \lambda)^{-1} \\
0 & (B - \lambda)^{-1}
\end{pmatrix}.
$$

If $\lambda \neq 0$ and the inverse of $T - \lambda$ exists, denote this inverse by the operator matrix $(\frac{X}{Y})$ a calculation shows that $Z = 0$, so that $W = (B - \lambda)^{-1}$.

**Lemma 1.** If $T$ is in $\mathcal{S}$, then $T$ is similar to an operator $M(A, B)$ such that $\|B\| < 1$.

**Proof.** If $T = M(A(T), B(T))$, Lemma 0 implies that $r(B(T)) < 1$, and Problem 122 of [3] implies that there exists an invertible operator $X$ such that $\|XB(T)X^{-1}\| < 1$. Since $T$ is similar to $M = M(A(T)X^{-1}, XB(T)X^{-1})$, the proof is complete.

**Lemma 2.** If $T$ is in $\mathcal{S}$ and $\text{nullity}(A(T)) = \text{corank}(A(T))$, then $T$ is similar to an operator $M(A, B)$ such that $A \geq 0$ and $\|B\| < 1$.

**Proof.** Consider the operator $M$ of Lemma 1. We have $\text{nullity}(A(T)X^{-1}) = \text{nullity}(A(T)) = \text{corank}(A(T)) = \text{corank}(A(T)X^{-1})$, and thus $(A(T)X^{-1} = UP$, where $U$ is unitary and $P \geq 0$. Since $M$ is unitarily equivalent to $M(P, XB(T)X^{-1})$, the proof is complete.

**Lemma 3.** Let $T = M(A, B)$ be in $\mathcal{S}$ and suppose $A^*A + B^*B \geq \epsilon^2 > 0$. If $|\lambda| > 1$, then $T$ is similar to $M(A - \epsilon/\lambda, B)$.

**Proof.** Theorem 1 of [1] implies that there exist operators $X_1$ and $X_2$ such that $X_1A + X_2B = \epsilon$ and $X_1^*X_1 + X_2^*X_2 \leq 1$. Let $|\lambda| > 1$ and let $S$ denote the operator on $\mathcal{H} \oplus \mathcal{H}$ whose matrix is

$$
\begin{pmatrix}
\frac{1}{\lambda^2} & \frac{1}{\lambda} \\
\frac{1}{\lambda} & 1
\end{pmatrix}.
$$
Now $S$ is invertible and a calculation shows that $SM(A, B)S^{-1}$ is of the desired form.

**Proof of Theorem 1-i.** The operator $M$ of Lemma 1 is similar to $M(XA(T)X^{-1}, XB(T)X^{-1})$, and thus we may assume that $\|B\| < 1$ and $0$ is not in the interior of $\sigma(A)$. By an application of Lemma 3 with $\lambda$ suitably chosen such that $A - \epsilon/\lambda$ is invertible and $|\lambda| > 1$, we may assume that $A$ is invertible. Since $\|B\| < 1$, we may define $R = A(1 - B^*B)^{-1/2}$ and $S = R \oplus I_\mathcal{C}$; a calculation shows that $S^{-1}TS = M((1 - B^*B)^{1/2}, B)$, which is a partial isometry, and therefore the proof is complete.

**Proof of Theorem 1-ii.** We may assume from Lemma 2 that $A \geq 0$; the result now follows from Theorem 1-i.

**Proof of Theorem 1-iii.** Recall that an operator $B$ in $\mathcal{L}(\mathcal{C})$ is not compact if and only if the range of $B$ contains a closed, infinite-dimensional subspace (see, for example, Theorem 2.5 of [2] and Problem 141 of [3]). It follows from this fact and an application of the open mapping theorem that $B$ is not compact if and only if $B$ is bounded below on some closed, infinite-dimensional subspace $M \subset \ker(B)^\perp$. Thus there exists $\delta > 0$ such that $\|Bm\| \leq \delta \|m\|$ for all $m$ in $M$. For each $m$ in $M$, we set $X_1(Bm) = Am$. Now

$$\|X_1(Bm)\| = \|Am\| \leq \|A\| \|m\| \leq (\|A\|/\delta) \|Bm\|,$$

and it follows that $X_1$ is a well-defined bounded linear operator defined on the closed subspace $B(M)$. Let $Q$ denote the projection onto $B(M)$, and let $X = X_1Q$ in $\mathcal{L}(\mathcal{C})$. Now $M \subset \ker(A - XB)$ and since $(A - XB)\mathcal{C} \subset A\mathcal{C}$, we have $\dim \ker(A - XB) = \dim \ker((A - XB)^*) = \mathcal{N}_0$. Since $T$ is similar to $M(A - XB, B)$, the proof may be completed by an application of Theorem 1-ii.

**Proof of Theorem 1-iv.** From Lemma 1, we may assume $\|B\| < 1$. Recall that an operator $B$ in $\mathcal{L}(\mathcal{C})$ is semi-Fredholm if $B$ has closed range and if either nullity($B$) or corank($B$) is finite. We consider first the case nullity($B$) $< \mathcal{N}_0$; there exists an operator $L$ and a finite rank operator $K$ such that $LB = 1 + K$. Let $X = (\sqrt{1 - B^*B} - A)L$ and let $S$ denote the operator on $\mathcal{H} \oplus \mathcal{C}$ whose matrix is $(\begin{smallmatrix} X & 1 \\ 0 & 1 \end{smallmatrix})$. A calculation shows that $STS^{-1} = M(\sqrt{1 - B^*B} + J, B)$, where $J$ is a finite rank operator. Since $\|B\| < 1$, $\sqrt{1 - B^*B} + J$ is Fredholm with index equal to zero, and the proof may be completed by an application of Theorem 1-ii.

We now consider the case corank($B$) $< \mathcal{N}_0$. In this case $B^*$ has finite nullity and closed range. Let $P$ denote the projection onto the initial space of $B^*$ and let $\mathcal{E} = \{x \in \mathcal{H} \mid \exists y \in P\mathcal{C} \text{ such that } A^*x + B^*y = 0\}$. Since $B^*$ has closed range, $\mathcal{E}$ is closed; since nullity $(T^*) = \mathcal{N}_0$ and nullity $(B^*) < \mathcal{N}_0$, $\mathcal{E}$ is infinite dimensional. For each $x$ in $\mathcal{E}$ there is a unique vector $X_1(x)$ in $P\mathcal{C}$ such that $A^*x + B^*X_1(x) = 0$. Since $B^*$ is bounded below on $P\mathcal{C}$, the assignment $x \mapsto X_1(x)$ is bounded and linear on the closed subspace $\mathcal{E}$. Let $Q$ denote the projection onto $\mathcal{E}$, and let $X = X_1Q$ in $\mathcal{L}(\mathcal{C})$; thus $\mathcal{E} \subset \ker(A + X^*B^*)^*$. Since $\mathcal{E}$ is infinite dimensional and $B$ is not compact, the proof may be completed by an application of Theorem 1-ii-iii.
Corollary. \( \mathcal{T} \subseteq \mathcal{P}^- \).

Proof. The preceding result implies that if \( T \) is in \( \mathcal{T} \) and \( B(T) \) is either left or right invertible, then \( T \) is in \( \mathcal{P} \). Now there exists a sequence \( \{B_k\} \subseteq \mathcal{L}(\mathcal{X}) \) such that \( \lim \|B_k - B(T)\| = 0 \) and such that the sequence elements are either all left invertible or all right invertible [3, Problem 109]. Since \( B_k^* B_k + A^* A \to B^* B + A^* A \), we may assume that each \( B_k^* B_k + A^* A \) is invertible; from the upper semicontinuity of the spectrum we may assume each \( r(B_k) < 1 \). Therefore, Theorem 1-iv implies that each \( M(A, B_k) \) is in \( \mathcal{P} \), and the proof is complete.

We now assume that \( T \) is in \( \mathcal{T} \) and that \( A^* \) has closed range and finite nullity. Let \( E \) be as in Theorem 1-v; the hypotheses imply that \( E \) is a closed, infinite-dimensional subspace. In view of the previous results it is natural to attempt to find an operator \( X \) such that \( \text{corank}(A + XB) = \aleph_0 \); the following result proves Theorem 1-v.

Proposition. There exists an operator \( X \) such that \( \text{corank}(A + XB) = \aleph_0 \) if and only if \( B^*|E \) is not compact.

Proof. If \( B^*|E \) is not compact, the operator \( X \) may be constructed by a straightforward modification of the proof of Theorem 1-iii; details are omitted.

For the converse, we assume that \( B^*|E \) is compact. Suppose that there is an operator \( X \) on \( \mathcal{X} \) and a closed, infinite-dimensional subspace \( K \subseteq \mathcal{X} \) such that \( A^* t = B^* X^* t \) for each \( t \) in \( K \). Since \( \dim \ker(A^*) < \aleph_0 \), it follows that \( L = K \cap \text{range}(A) \) is infinite dimensional. Since \( A^* \) has closed range, \( A^* \) is bounded below on \( L \). Let \( \{t_n\} \) denote an orthonormal basis for \( L \). Now \( t_n \to 0 \), \( \{X^*(t_n)\} \subseteq E \), and thus \( B^* X^* t_n \to 0 \). Therefore \( A^* t_n \to 0 \), which is a contradiction.

Proof of Theorem 2-i. Let \( A = UP \) denote the polar decomposition of \( A \). Since \( P^2 + B^* B \) is invertible, we may define \( T_1 = M(P, B) \), and Lemma 0 implies that \( r(T_1) = r(B) = r(M(A, B)) < 1 \). Theorem 1-ii now implies that \( T_1 \) is similar to a partial isometry. Since the nullity and corank of \( U \) are finite, \( U \) is unitary, and the proof is completed by noting that

\[
T_1 - (U^* \oplus 1)T(U \oplus 1)
\]

is of finite rank.

Proof of Theorem 2-ii. Theorem 1 of [1] implies that there exist operators \( X_1 \) and \( X_2 \) such that \( X_1 A + X_2 B = 1 \). Since \( B \) is compact, we have \( \tilde{X}_1 \tilde{A} = 1 \), and thus \( A \) has closed range and finite nullity. If \( A = UP \) denotes the polar decomposition of \( A \), then \( P = Q \oplus 0 \), where \( Q \) is invertible. Set \( R = Q^{-1} \oplus 1_\ker(P) \) and \( S = 1_\mathcal{X} \oplus R \). Now \( S^{-1} TS \) has the operator matrix

\[
\begin{pmatrix}
0 & U \\
0 & R^{-1}BR
\end{pmatrix},
\]

which is the sum of a partial isometry and a compact operator.
REFERENCES


Department of Mathematics, Western Michigan University, Kalamazoo, Michigan 49001