WHICH OPERATORS ARE SIMILAR TO PARTIAL ISOMETRIES?

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Abstract. Let $\mathcal{H}$ denote a separable, infinite dimensional complex Hilbert space and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{P} = \{ T \in \mathcal{B}(\mathcal{H}) | r(T) < 1 \text{ and } T \text{ is similar to a partial isometry with infinite rank} \}$; let $\mathcal{S} = \{ S \in \mathcal{B}(\mathcal{H}) | r(S) < 1, \text{range}(S) \text{ is closed, and rank}(S) = \text{nullity}(S) = \text{corank}(S) = \aleph_0 \}$. It is conjectured that $\mathcal{P} = \mathcal{S}$ and it is proved that $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{P}^\gamma$.

Introduction. Let $\mathcal{H}$ denote a fixed separable, infinite-dimensional complex Hilbert space, and let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. In [5], Sz.-Nagy proved that an invertible operator $T$ in $\mathcal{B}(\mathcal{H})$ is similar to a unitary operator if and only if the powers of $T$ and $T^{-1}$ are uniformly bounded; the proof of this result also implies that an operator is similar to an isometry if and only if its powers are uniformly bounded above and below [4]. In this note we state the following conjecture concerning operators similar to partial isometries, and then prove results which partially affirm the conjecture.

Conjecture. If $T$ is an operator on $\mathcal{H}$ with closed range, whose spectral radius is less than one, and such that $\text{rank}(T) = \text{nullity}(T) = \text{nullity}(T^*) = \aleph_0$, then $T$ is similar to a partial isometry.

Let $\mathcal{P} = \{ T \in \mathcal{B}(\mathcal{H}) | r(T) < 1 \text{ and } T \text{ is similar to a partial isometry with infinite rank} \}$, where $r(T)$ is the spectral radius of $T$; let $\mathcal{S} = \{ S \in \mathcal{B}(\mathcal{H}) | r(S) < 1, \text{range}(S) \text{ is closed, and rank}(S) = \text{nullity}(S) = \text{corank}(S) = \aleph_0 \}$. It is easy to prove that $\mathcal{P} \subseteq \mathcal{S}$ and in this note we prove that $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{P}^\gamma$ (the norm closure of $\mathcal{P}$ in $\mathcal{B}(\mathcal{H})$). To state the results in detail we use the following notation. If $A$ and $B$ are operators on $\mathcal{H}$ such that $A^*A + B^*B$ is invertible, let $M(A, B)$ denote the operator on $\mathcal{H} \oplus \mathcal{H}$ whose matrix is $(A, B)$; let $\mathcal{S}$ denote the set of all matrices of this form whose spectral radii are less than one. Each operator in $\mathcal{S}$ is unitarily equivalent to a matrix in $\mathcal{S}$.

Theorem 1. The operator $M(A, B)$ in $\mathcal{S}$ is similar to a partial isometry if any of the following conditions are satisfied:

(i) 0 is not in the interior of $\sigma(A)$;
(ii) $\text{nullity}(A) = \text{corank}(A)$;
(iii) $\text{nullity}(A) < \text{corank}(A) = \aleph_0$ and $B$ is not compact;
(iv) $B$ is a semi-Fredholm operator;
(v) $\text{corank}(A) < \text{nullity}(A)$, $A$ has closed range, and $B^*|E$ is not compact, where $E = \{ y \in \mathcal{H} | \exists x \in \mathcal{H} : A^*x + B^*y = 0 \}$.

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Let \((J)\) denote the ideal of all compact operators in \(\mathcal{L}(\mathcal{H} \oplus \mathcal{H})\). If \(T\) is in \(\mathcal{L}(\mathcal{H} \oplus \mathcal{H})\), let \(\hat{T}\) denote the image of \(T\) under the canonical homomorphism of \(\mathcal{L}(\mathcal{H} \oplus \mathcal{H})\) onto the Calkin algebra \(\mathcal{L}(\mathcal{H} \oplus \mathcal{H})/(J)\).

**Theorem 2.** If \(T = M(A, B)\) is in \(\mathcal{T}\) then \(\hat{T}\) is similar to a partial isometry if either of the following conditions is satisfied:

(i) \(\text{nullity}(A)\) and \(\text{corank}(A)\) are finite;

(ii) \(B\) is compact.

Because these results do not cover the case \(\text{corank}(A) < \text{nullity}(A) = \mathcal{N}_0\), we do not know whether \(\mathcal{T} = \mathcal{S}\). We note also that the proof of Theorem 1-iii was motivated by the proof of a factorization theorem of R. G. Douglas [1, Lemma 2.1]. The author thanks the referee for suggestions that have clarified certain points in the original proofs of our results.

**Proof of Theorems 1 and 2.**

**Lemma 0.** If \(T = M(A, B)\) is in \(\mathcal{T}\), then the nonzero elements of \(\sigma(T)\) and \(\sigma(B)\) are identical.

**Proof.** If \(\lambda \neq 0\) and \(B - \lambda\) is invertible, then a calculation shows that \((T - \lambda)^{-1}\) is given by the operator matrix

\[
\begin{pmatrix}
-1/\lambda & (-1/\lambda)A(B - \lambda)^{-1} \\
0 & (B - \lambda)^{-1}
\end{pmatrix}
\]

If \(\lambda \neq 0\) and the inverse of \(T - \lambda\) exists, denote this inverse by the operator matrix \((X, Y)\); a calculation shows that \(Z = 0\), so that \(W = (B - \lambda)^{-1}\).

**Lemma 1.** If \(T\) is in \(\mathcal{T}\), then \(T\) is similar to an operator \(M(A, B)\) such that \(\|B\| < 1\).

**Proof.** If \(T = M(A(T), B(T))\), Lemma 0 implies that \(r(B(T)) < 1\), and Problem 122 of [3] implies that there exists an invertible operator \(X\) such that \(\|XB(T)X^{-1}\| < 1\). Since \(T\) is similar to \(M = M(A(T)X^{-1}, XB(T)X^{-1})\), the proof is complete.

**Lemma 2.** If \(T\) is in \(\mathcal{T}\) and \(\text{nullity}(A(T)) = \text{corank}(A(T))\), then \(T\) is similar to an operator \(M(A, B)\) such that \(A \geq 0\) and \(\|B\| < 1\).

**Proof.** Consider the operator \(M\) of Lemma 1. We have \(\text{nullity}(A(T)X^{-1}) = \text{nullity}(A(T)) = \text{corank}(A(T)) = \text{corank}(A(T)X^{-1})\), and thus \(A(T)X^{-1} = UP\), where \(U\) is unitary and \(P \geq 0\). Since \(M\) is unitarily equivalent to \(M(P, XB(T)X^{-1})\), the proof is complete.

**Lemma 3.** Let \(T = M(A, B)\) be in \(\mathcal{T}\) and suppose \(A^*A + B^*B \geq \varepsilon^2 > 0\). If \(|\lambda| > 1\), then \(T\) is similar to \(M(A - \varepsilon/\lambda, B)\).

**Proof.** Theorem 1 of [1] implies that there exist operators \(X_1\) and \(X_2\) such that \(X_1A + X_2B = \varepsilon\) and \(X_1^*X_1 + X_2^*X_2 < 1\). Let \(|\lambda| > 1\) and let \(S\) denote the operator on \(\mathcal{H} \oplus \mathcal{H}\) whose matrix is
Now $S$ is invertible and a calculation shows that $SM(A,B)S^{-1}$ is of the desired form.

**Proof of Theorem 1-i.** The operator $M$ of Lemma 1 is similar to $M(XA(T)X^{-1}, XB(T)X^{-1})$, and thus we may assume that $\|B\| < 1$ and 0 is not in the interior of $\sigma(A)$. By an application of Lemma 3 with $\lambda$ suitably chosen such that $A - \epsilon/\lambda$ is invertible and $|\lambda| > 1$, we may assume that $A$ is invertible. Since $\|B\| < 1$, we may define $R = A(1 - B*B)^{-1/2}$ and $S = R \oplus I_K$; a calculation shows that $S^{-1}TS = M((1 - B*B)^{1/2}, B)$, which is a partial isometry, and therefore the proof is complete.

**Proof of Theorem 1-ii.** We may assume from Lemma 2 that $A \geq 0$; the result now follows from Theorem 1-i.

**Proof of Theorem 1-iii.** Recall that an operator $B$ in $\ell(\mathcal{C})$ is not compact if and only if the range of $B$ contains a closed, infinite-dimensional subspace (see, for example, Theorem 2.5 of [2] and Problem 141 of [3]). It follows from this fact and an application of the open mapping theorem that $B$ is not compact if and only if $B$ is bounded below on some closed, infinite-dimensional subspace $M \subset \ker(B)$. Thus there exists $\delta > 0$ such that $\|Bm\| \geq \delta \|m\|$ for all $m$ in $M$. For each $m$ in $M$, we set $X_1(Bm) = Am$. Now

$$\|X_1(Bm)\| = \|Am\| \leq \|A\| \|m\| \leq (\|A\|/\delta) \|Bm\|,$$

and it follows that $X_1$ is a well-defined bounded linear operator defined on the closed subspace $B(M)$. Let $Q$ denote the projection onto $B(M)$, and let $X = X_1Q$ in $\ell(\mathcal{C})$. Now $M \subset \ker(A - XB)$ and since $(A - XB)\mathcal{C} \subset A\mathcal{C}$, we have $\dim \ker(A - XB) = \dim \ker((A - XB)^*) = \mathfrak{N}_0$. Since $T$ is similar to $M(A - XB, B)$, the proof may be completed by an application of Theorem 1-ii.

**Proof of Theorem 1-iv.** From Lemma 1, we may assume $\|B\| < 1$. Recall that an operator $B$ in $\ell(\mathcal{C})$ is semi-Fredholm if $B$ has closed range and if either nullity($B$) or corank($B$) is finite. We consider first the case nullity($B$) $< \mathfrak{N}_0$; there exists an operator $L$ and a finite rank operator $K$ such that $LB = 1 + K$. Let $X = (\sqrt{1 - B*B} - A)L$ and let $S$ denote the operator on $\mathcal{C} \oplus \mathcal{C}$ whose matrix is $(I \; X)$. A calculation shows that $STS^{-1} = M(\sqrt{1 - B*B} + J, B)$, where $J$ is a finite rank operator. Since $\|B\| < 1$, $\sqrt{1 - B*B} + J$ is Fredholm with index equal to zero, and the proof may be completed by an application of Theorem 1-ii.

We now consider the case corank($B$) $< \mathfrak{N}_0$. In this case $B^*$ has finite nullity and closed range. Let $P$ denote the projection onto the initial space of $B^*$ and let $\mathcal{E} = \{x \in \mathcal{C} | \exists y \in P\mathcal{C} \text{ such that } A^*x + B^*y = 0\}$. Since $B^*$ has closed range, $\mathcal{E}$ is closed; since nullity($T^*$) $= \mathfrak{N}_0$ and nullity($B^*$) $< \mathfrak{N}_0$, it follows readily that $\mathcal{E}$ is infinite dimensional. For each $x$ in $\mathcal{E}$ there is a unique vector $X_1(x)$ in $P\mathcal{C}$ such that $A^*x + B^*X_1(x) = 0$. Since $B^*$ is bounded below on $P\mathcal{C}$, the assignment $x \rightarrow X_1(x)$ is bounded and linear on the closed subspace $\mathcal{E}$. Let $Q$ denote the projection onto $\mathcal{E}$ and let $X = X_1Q$ in $\ell(\mathcal{C})$; thus $\mathcal{E} \subset \ker(A + X'B^*)$. Since $\mathcal{E}$ is infinite dimensional and $B$ is not compact, the proof may be completed by an application of Theorem 1-ii-iii.
Corollary. \(\mathcal{T} \subset \mathcal{P}^-\).

Proof. The preceding result implies that if \(T\) is in \(\mathcal{T}\) and \(B(T)\) is either left or right invertible, then \(T\) is in \(\mathcal{P}\). Now there exists a sequence \(\{B_k\} \subset \mathcal{L}(\mathcal{H})\) such that \(\lim \|B_k - B(T)\| = 0\) and such that the sequence elements are either all left invertible or all right invertible [3, Problem 109]. Since \(B_k^* B_k + A^* A \to B^* B + A^* A\), we may assume that each \(B_k^* B_k + A^* A\) is invertible; from the upper semicontinuity of the spectrum we may assume each \(r(B_k) < 1\). Therefore, Theorem 1-iv implies that each \(M(A, B_k)\) is in \(\mathcal{P}\), and the proof is complete.

We now assume that \(T\) is in \(\mathcal{T}\) and that \(A^*\) has closed range and finite nullity. Let \(E\) be as in Theorem 1-v; the hypotheses imply that \(E\) is a closed, infinite-dimensional subspace. In view of the previous results it is natural to attempt to find an operator \(X\) such that \(\text{corank}(A + XB) = \mathfrak{N}_0\); the following result proves Theorem 1-v.

Proposition. There exists an operator \(X\) such that \(\text{corank}(A + XB) = \mathfrak{N}_0\) if and only if \(B^*|E\) is not compact.

Proof. If \(B^*|E\) is not compact, the operator \(X\) may be constructed by a straightforward modification of the proof of Theorem 1-iii; details are omitted.

For the converse, we assume that \(B^*|E\) is compact. Suppose that there is an operator \(X\) on \(\mathcal{H}\) and a closed, infinite-dimensional subspace \(K \subset \mathcal{H}\) such that \(A^* t = B^* X^* t\) for each \(t\) in \(K\). Since \(\dim \ker(A^*) < \mathfrak{N}_0\), it follows that \(L = K \cap \text{range}(A)\) is infinite dimensional. Since \(A^*\) has closed range, \(A^*\) is bounded below on \(L\). Let \(\{t_n\}\) denote an orthonormal basis for \(L\). Now \(t_n \to 0\), \(\{X^* (t_n)\} \subset E\), and thus \(B^* X^* t_n \to 0\). Therefore \(A^* t_n \to 0\), which is a contradiction.

Proof of Theorem 2-i. Let \(A = UP\) denote the polar decomposition of \(A\). Since \(P^2 + B^* B\) is invertible, we may define \(T_1 = M(P, B)\), and Lemma 0 implies that \(r(T_1) = r(B) = r(M(A, B)) < 1\). Theorem 1-ii now implies that \(T_1\) is similar to a partial isometry. Since the nullity and corank of \(U\) are finite, \(\tilde{U}\) is unitary, and the proof is completed by noting that

\[
T_1 - (U^* \oplus 1)T(U \oplus 1)
\]

is of finite rank.

Proof of Theorem 2-ii. Theorem 1 of [1] implies that there exist operators \(X_1\) and \(X_2\) such that \(X_1 A + X_2 B = 1\). Since \(B\) is compact, we have \(\tilde{X}_1 \tilde{A} = 1\), and thus \(A\) has closed range and finite nullity. If \(A = UP\) denotes the polar decomposition of \(A\), then \(P = Q \oplus 0\), where \(Q\) is invertible. Set \(R = Q^{-1} \oplus 1_{\ker(P)}\) and \(S = I_{\mathcal{H}} \oplus R\). Now \(S^{-1} TS\) has the operator matrix

\[
\begin{pmatrix}
0 & U \\
0 & R^{-1} BR
\end{pmatrix},
\]

which is the sum of a partial isometry and a compact operator.
REFERENCES


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