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RESIDUAL EQUISINGULARITY

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ABSTRACT. Let V be a complex analytic set and $\mathrm{Sg}\,V$ the singular set of V be in codimension one; then the set of points of $\mathrm{Sg}\,V$ for which V is not residually equisingular along $\mathrm{Sg}\,V$ is a proper analytic subset of $\mathrm{Sg}\,V$. V is said to be residually equisingular along $\mathrm{Sg}\,V$ if all one dimensional slices of V transverse to $\mathrm{Sg}\,V$ have isomorphic resolutions.

Let V be an analytic subset of a domain $U \subset \mathbf{C}^n$ of pure dimension r and let $\mathrm{Sg}\,V$, the singular locus of V, be of dimension $r - 1$. For $p \in \mathrm{Sg}\,V$ such that near p, $\mathrm{Sg}\,V$ is an $(r - 1)$-dimensional manifold, the second author introduced [3] three different notions of equisingularity of V along $\mathrm{Sg}\,V$ at p: weak, strong, and residual. These generalize various portions of the theory of equisingularity for hypersurfaces developed by Zariski [5]. The results pertaining to residual equisingularity were published jointly in [1]. The purpose of this note is to improve one of the results in [1]. There we showed that if V is weakly equisingular along $\mathrm{Sg}\,V$ at p, then near p $\{q \in \mathrm{Sg}\,V$: V is not residually equisingular along $\mathrm{Sg}\,V$ at $q\}$ is contained in an analytic subset of $\mathrm{Sg}\,V$ of dimension $< r - 1$. We improve this showing that $\{q \in \mathrm{Sg}\,V$: V is not residually equisingular along $\mathrm{Sg}\,V$ at $q\}$ is an analytic subset of $\mathrm{Sg}\,V$ of dimension $< r - 1$. In the course of our proof we will clarify our original definition of residual equisingularity and develop an alternative formulation which does not depend on the imbedding of V in U.

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Let p, $\mathrm{Sg}\,V$ and V be as above. Replacing U by a suitable neighborhood of p we may select coordinates (x_i) such that p becomes the origin, $\mathrm{Sg}\,V = \{q \in U: q_i = 0 \text{ for } i \geqslant r\}$, and for $q \in \mathrm{Sg}\,V$, $V_q = \{x \in V: x_i = q_i \text{ for } i < r\}$ is one dimensional. Such coordinates will be said to be adapted to V at p. Each V_q can be resolved by a sequence of local quadratic transformations, however the series may vary greatly with q; V is said to be residually equisingular (see Definitions 1 and 2) if the series does not vary. For such coordinates, and for W, W' appropriate open subsets of V, we define 3 types of maps from W to W':

(i) $T(x) = (x_1, \ldots, x_r, x_{r+1} - t_1(x), \ldots, x_n - t_{n-r}(x))$ where the t_i are holomorphic functions of $x_1, \ldots, x_r$ alone.

(ii) $S(x) = (x_1, \ldots, x_{r-1}, s_0(x), \ldots, s_{n-r}(x))$ where $s_i(x) = \sum_{j=r}^{n} a_{ij} x_j$, $a_{i,j} \in \mathbf{C}$, and $rk(a_{i,j}) = n - r$.

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(iii) $Q(x) = (x_1, \ldots, x_r, x_r x_{r+1}, \ldots, x_r x_n)$.

DEFINITION 1. A connected sequence of local quadratic transformations is a sequence of maps f_j: $W_j \to W_{j-1}$, $i = 1, \ldots, l$, such that:

(a) $F_j = f_1 \circ f_2 \circ \cdots \circ f_j$ is a proper modification of W_0, that is F_j^{-1} (compact set) is compact and F_j is biholomorphic on the inverse image of Reg W_0, the regular points of W_0, and $F_j^{-1}(\text{Reg } W_0)$ is dense in W_j.

(b) f_j is the restriction of a map of type (i), (ii) or (iii) on each component $W_{j,k}$ of W_j.

(c) If $f_j(W_{j,k_1}) \cap f_j(W_{j,k_2})$ is nonnull then $f_j | W_{j,k_1}$ and $F | W_{j,k_2}$ are of the same type.

Suppose that a connected sequence of quadratic transformations is given and the image of F_j is a neighborhood of p. Set $V_j = F_j^{-1}(V)$ (of course if a type (iii) map occurs, we omit those components which are contained in $\{x \in U: x_r = 0\}$).

DEFINITION 2. V is residually equisingular along Sg V at p if there is a connected sequence of quadratic transformations as above such that:

(a) F_l: $V_l \to V$ is a resolution;

(b) $F_l^{-1}(\text{Sg } V)$ is an $r-1$ manifold and F_l restricted to each component of it is biholomorphic.

The above definition appears weaker than that given in [1]. There we required that given any $q \in F_{j_0}^{-1}(p)$ and $V_{j_0,k}$ a branch of V_{j_0} at q that either $V_{j_0,k}$ is nonsingular at q or Sg $V_{j_0,k} = F_{j_0}^{-1}(\text{Sg } V)$ near q. This condition is actually a consequence of our other conditions. To see this assume there is a q at which Sg $V_{j_0,k}$ is a proper subset of $F_{j_0}^{-1}(\text{Sg } V)$; then $\dim_q$ Sg $V_{j_0,k} < r-1$. Since the maps $(f_j)^{-1}$ are weakly holomorphic the singular loci of the portions of the V_j, $j > j_0$, lying over $V_{j_0+1,k}$ are also of dimension $< r-1$. Thus it suffices to consider the case where $V_{j_0+1,k}$ is nonsingular and $f_{j_0+1} | V_{j_0+1,k}$ is of type (iii). Now as in the first portion of the proof of Proposition 1 in [1], one sees that dim $C_4(V_{j_0+1,k}, q) = r$; and [2, Proposition 1.8] dim $C_4 V = \dim V$, codim Sg $V \geqslant 2$ implies V nonsingular.

DEFINITION 3. Let X be an analytic variety and $\mathcal{J}$ a coherent sheaf of ideals in ${}_X\mathcal{O}$, the reduced sheaf of homolomorphic functions on X, ${}_n\mathcal{O}/I(V)$ where I is self-radical. Then the blow-up of X along $\mathcal{J}$, $B_J(X)$, is defined as follows: let $g_1, \ldots, g_m$ generate $\mathcal{J}$ over ${}_X\mathcal{O}$, $Y = \text{support}\,\mathcal{J}$, and $B_J(X)$ the closure in $X \times \mathbf{CP}^{m-1}$ of $\{(x, z) \in (X - Y) \times \mathbf{CP}^{m-1}: z_i g_j(x) = z_j g_i(x) \text{ all } i, j\}$. Then the natural projection π: $B_J(X) \to X$ is a proper modification of X. This construction is independent (up to analytic isomorphism) of the choice of generators of $\mathcal{J}$ and of the coordinates on $\mathbf{C}^n$.

Now let $\mathcal{J}$ be the ideal sheaf of SgX and $B_s(X)$ denote the blow-up of X along $\mathcal{J}$. We will see there is a one-to-one correspondence between local quadratic transforms of X and blow-ups such that $\pi^{-1}(\text{Sg}(X)) \to \text{Sg}\,X$ is a biholomorphism on each component.

Let $p \in$ Reg (Sg (X)), (x_i) coordinates adapted to X at p, and Q: $X' \to X$ a local quadratic transform of the third type $Q(x_1, \ldots, x_n) = (x_1, \ldots, x_r, x_r x_{r+1}, \ldots, x_r x_n) = (y_1, \ldots, y_n)$. Now $I(\text{Sg } V)$ is generated by $x_r, x_{r+1}, \ldots, x_n$, so form the blow-up $B_s(X) \subset \{(y, z) \in X \times \mathbf{CP}^{n-r}: y_i z_j = y_j z_i \text{ for } i, j \geqslant r\}$. There is an injection i: $X' \to B_s(X)$,

$$i(x_1, \ldots, x_n) = (x_1, \ldots, x_r, x_r x_{r+1}, \ldots, x_r x_n; 1, x_{r+1}, \ldots, x_n)$$

such that $Q = \pi i$. Since $B_s(X) \subset \{(y,z): z_r \neq 0\}$, β: $B_s(X) \to X'$, $\beta(x,y) = x$ is the inverse to i. Since Q is unramified over $\operatorname{Sg} X$, π is also.

Conversely if π: $B_s(X) \to X$ is unramified over $\operatorname{Sg} X$, one can define an inverse φ: $\operatorname{Sg} X \to B_s(X)$ of π and by shrinking the neighborhood of p, assume that some Z_i (say Z_r) is nonzero on the image of φ. Let

$$Q(y_1,\ldots,y_n) = (y_1,\ldots,y_r,y_r y_{r+1},\ldots,y_r y_n)$$

and $X' = Q^{-1}(X)$; there is an analytic equivalence ρ: $B_s(X) \to X'$ defined by

$$\rho(x_1,\ldots,x_n,z_r,\ldots,z_n) = (x_1,\ldots,x_r,z_{r+1}/z_r,\ldots,z_n/z_r)$$

such that $\pi = Q\rho$. Q is unramified over $\operatorname{Sg} V$, since π is too.

For each integer $j > 0$ we can define a space B_j and a map g_j: $B_j \to V$ as follows: π_1: $B_1 \to V$ is the blow-up of V along $\operatorname{Sg} V$. For $j > 1$, set $S_j = g_j^{-1}(\operatorname{Sg} V)$, let π_j: $B_{j+1} \to B_j$ be the blow-up of B_j along S_j, and set $g_{j+1} = \pi_j \circ g_j$.

Lemma. *The following are equivalent*:

(a) *V is residually equisingular along* $\operatorname{Sg} V$ *at* $p \in \operatorname{Sg} V$.

(b) *If one replaces U by a suitable neighborhood of p, then there is an integer $m > 0$ such that B_m is nonsingular, and for $j \leqslant m$, S_j is an $(r-1)$-dimensional manifold, and π_j restricted to each component of S_j is biholomorphic.*

Proof. Follows immediately from the above and the fact that types (i) and (ii) maps are biholomorphic.

Proposition. *The set A of points p such that V is not residually equisingular along* $\operatorname{Sg} V$ *at p is an analytic subset of V.*

Proof. Take a succession of blow-ups of V along the singular locus; define inductively $V_0 = V$, $V_{i+1} = B_{\operatorname{Sg} V_i}(V_i)$, π_i: $V_i \to V_{i-1}$, $g_j = \pi_1 \circ \cdots \circ \pi_j$ and $S_i = g_i^{-1}(\operatorname{Sg} V)$.

Since A is a nowhere dense set in $\operatorname{Sg} V$ [1, Proposition 2], there exists some $l > 0$ such that $\operatorname{codim}_{V_l} \operatorname{Sg} V_l \geqslant 2$. Then, we have that

$$A = g_l(\operatorname{Sg} B_l) \cup \left(\bigcup_j g_j(\operatorname{Sg} S_j \cup \{q \in S_j - \operatorname{Sg} S_j: rk_q g_j|S_j < r-1\}) \right)$$

is the finite union of images of analytic sets under proper maps.

In addition to providing a characterization of residual equisingularity, the Lemma also provides the justification for a simple method of testing for residual equisingularity in particular examples. As observed in [3], since residual equisingularity implies weak equisingularity, all potential equisingular points have Puiseux series normalizations, so one can attempt to check for residual equisingularity by direct calculation. For example let V be the image of the map f: $\mathbf{C}^2 \to \mathbf{C}^4$ given by $f(x,y) = (x,y^9,y^7,xy^8)$. (See [1, Example 2] for more details on this example.) The obvious attempt to construct the required connected sequence of local quadratic transformations leads to a dead end. However, initially one does not know that some other sequence will not work. By constructing from the initial sequence the blow-ups B_1, B_2, one sees that $\dim \pi_2^{-1}(0) = 1$, so V is not residually equisingular at 0.

In the classical theory of plane curves, one has the equivalence of characteristic pairs and multiplicities e_i of the canonical sequence of blow-ups which resolve the singularity of an irreducible plane curve. These determine the length of the conductor $\dim_{\mathbf{C}} \tilde{\mathcal{O}}/\mathcal{O}$, and the, exponent of the conductor on the normalization $\dim_{\mathbf{C}} \tilde{\mathcal{O}}/\mathcal{O}$, via the Italian Geometers formula:

$$\exp = 2 \cdot \text{length} = \sum e_i(e_i - 1).$$

For equisingularity in codim one [5], these results extend as follows: V is equisingular along $\operatorname{Sg} V$ if and only if the curves have the same multiplicity sequence. Furthermore, if the length of the conductor is the same for each curve, then V is equisingular along $\operatorname{Sg} V$.

For equisingularity in higher codim, this all breaks down. Neither the exponent or length is constant, and it is not true that one thing is twice the other.

EXAMPLE. Let V be the image in $\mathbf{C}^4$ of $(s,t) \to (s,t^3,t^4,st^5)$. Then V is residually and strongly equisingular. For $s \neq 0$, exponent $= 3$, length $= 2$, and $\sum e_i(e_i - 1) = 6$. For $s = 0$, exponent $= 6$, length $= 3$, and $\sum e_i(e_i - 1) = 6$.

However, one can easily see from looking at the Puiseux expansions the multiplicity sequence is the same for each curve in a residually equisingular variety and so $\sum e_i(e_i - 1)$ is constant. Recently Fischer [4] has found a possible interpretation for this number: let C be the complete local ring of a irreducible curve, $\bar{C}$ its integral closure in its field of quotients, and I_C the high order differentials of $\bar{C}$ over C. Then length $I_C = \sum e_i(e_i - 1)$ and $I_{C_1} \simeq I_{C_2}$ as $\bar{C}$ modules if and only if C_1 and C_2 have the same multiplicity sequence.

REMARK. Strong equisingularity does not even imply that the multiplicity sequence is constant. This can be seen from the example $(s,t) \to (s,t^5,t^7,st^8)$.

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