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ON THE STRAIGHTNESS OF REDUCED TEICHMÜLLER SPACE

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ABSTRACT. Under a natural injection which is shown to be an isometry, the image of the reduced Teichmüller space $T^{\#}(G)$ in the open straight Teichmüller space of the Fuchsian model of G, $T(H)$, is an open straight subspace.

It is known that, with the Teichmüller metric, the Teichmüller space $T(H)$ of a nonelementary finitely generated Fuchsian group H of the first kind is an open straight space in the sense of Busemann [9], [10]. Hence, between any two distinct points of $T(H)$ there is a unique geodesic locally isometric to $\mathbf{R}$. This paper considers an extension of this property to $T^{\#}(G)$, the reduced Teichmüller space of a finitely generated nonelementary Fuchsian group G of the second kind.

Let $M_1(H)$ $(M_1(G))$ be the set of Beltrami differentials $\mu(z)d\bar{z}/dz$ on the upper half plane U satisfying $\|\mu\| = \text{ess sup}|\mu(z)| < 1$, $\mu(h(z))\overline{h'(z)}/h'(z) = \mu(z)$ for all $h \in H$ $(g \in G)$.

Extend $\mu(z)$ to $\mathbf{C}$ by $\mu(\bar{z}) = \bar{\mu}(z)$, and let $w_\mu(z)$ be the unique solution of the Beltrami differential equation $w_{\bar{z}} = \mu w_z$ fixing 0,1, and ∞. The Teichmüller space $T(H)$ $(T(G))$ is the set of equivalence classes $[w_\mu]$ of elements of $M_1(H)$ $(M_1(G))$ where $\mu \sim \nu$ if and only if $w_\mu = w_\nu$ on $\mathbf{R}$. The reduced Teichmüller space is the set of equivalence classes θ_μ of elements of $M_1(G)$ where $\mu \underset{G}{\sim} \nu$ if and only if $w_\mu = w_\nu$ on $\Lambda(G)$, the limit set of G. This is in turn equivalent to the condition that the induced isomorphisms $g \to w_\mu g w_\mu^{-1}$ and $g \to w_\nu g w_\nu^{-1}$ are identical. (Note that for H of the first kind $T^{\#}(H) = T(H)$.)

Since G is a finitely generated Fuchsian group of the second kind, $\Lambda(G) \subsetneq \mathbf{R}$, and we have $U \xrightarrow{\rho} \Omega(G) \xrightarrow{\pi} \Omega(G)/G$ where ρ is a holomorphic cover map, π is a (possibly ramified) holomorphic cover, $\Omega(G)$ is the ordinary set of G, and $\Omega(G)/G$ is the double of U/G. Let J: $U \to U$ be given by $J(z) = -\bar{z}$; the cover ρ may be chosen to satisfy $\rho(J(z)) = \bar{\rho}(z)$. Fix H by defining $H = \{h \in PSL(2,\mathbf{R}) | \rho \circ h = g \circ \rho \text{ for some } g \in G\}$.

Let $B_2(H,U)$ be the set of quadratic differentials with respect to H on U satisfying $\|\phi\| = \sup|\phi(z)y^2| < \infty$, and let $B_2'(H,U) = \{\phi \in B_2(H,U) | \phi(Jz) = \bar{\phi}(z)\}$. Let $B_2(G,\Omega)$ be those quadratic differentials on $\Omega(G)$ which are real

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on $\Omega(G) \cap \mathbf{R}$ and satisfy $\|\psi\| = \sup|(\psi \cdot \lambda^{-2})(z)| < \infty$, where λ is the Poincaré metric induced by $\pi \circ \rho$.

Finally, let $M_1'(H) = \{\eta \in M_1(H) | \eta(J(z)) = \bar{\eta}(z)\}$; this is equivalent to the condition that w_η commutes with J. Via the isometries $\mu(z) \to \rho \cdot \mu(z) = \mu(\rho(z))(\overline{\rho'(z)}/\rho'(z))$ and $\psi(z) \to (\psi \times \rho)(z) = \psi(\rho(z))(\rho'(z))^2$, it can be easily demonstrated that $M_1(G)$ is isomorphic to $M_1'(H)$, and $B_2(G, \Omega)$ is isomorphic to $B_2'(H, U)$. Earle has shown [7], [8] that the former induces the map $\theta_\mu \to [w_{\rho \cdot \mu}]$ which is a real analytic injection of $T^{\#}(G)$ into $T(H)$ with image $T'(H) = \{[w_{\hat{\eta}}] \in T(H) | \eta \in M_1'(H)$ for some $\eta \sim \hat{\eta}\}$.

1. **The straightness of** $T'(H)$. We wish to study further the structure of $T'(H)$. It is clear that if $\mu \in [w_{\hat{\mu}}] \in T'(H)$, w_μ commutes with J on $\mathbf{R}$ (i.e., is odd).

LEMMA 1. *The unique extremal element w_η in each equivalence class $[w_{\hat{\eta}}] \in T(H)$ with odd boundary values is symmetric ($w_\eta \circ J(z) = J \circ w_\eta(z)$ on U), and $T'(H) = \{[w_\mu] \in T(H) | w_\mu$ is odd on $\mathbf{R}\}$.*

PROOF. Suppose $w = w_\eta$ is not symmetric, and set $f(z) = (J \circ w \circ J)(z)$. Then the complex dilatation η_1 of f is $(\bar{\eta} \circ J)(z)$. Since J is a homeomorphism, $\|\eta\| = \|\bar{\eta} \circ J\|$. Since w_η is odd on $\mathbf{R}$, $f(x) = w_\eta(x)$, and $f \sim w_\eta$. Since $\|\eta_1\| = \|\eta\|$, by Teichmüller's theorem for $T(H)$, $J \circ w_\eta \circ J = f = w_\eta$, and so w_η commutes with J on U. Hence $\eta \in M_1'(H)$, and equivalence classes of $T(H)$ with odd boundary values are contained in $T'(H)$. The other inclusion is obvious. □

Let w_η be the extremal element of $[w_{\hat{\eta}}] \in T'(H)$. Since $M(G)$ is isomorphic to $M'(H)$, $w_\eta = w_{\rho \cdot \alpha}$ for some $\alpha \in M_1(G)$. Now α must be uniquely extremal, for if $\beta \in M_1(G)$, $\alpha \underset{G}{\sim} \beta$, and $\|\beta\| \leqslant \|\alpha\|$, then $\rho \cdot \beta \sim \rho \cdot \alpha$ with $\|\rho \cdot \beta\| \leqslant \|\rho \cdot \alpha\| = \|\eta\|$, which is a contradiction. Hence

PROPOSITION 1. *Each equivalence class $\theta_{\hat{\mu}}$ of $T^{\#}(G)$ contains a unique extremal element w_μ; $w_{\rho \cdot \mu}$ is the extremal element of $[w_{\rho \cdot \mu}] \in T'(H)$.*

Since we now have an extremal element in each class, the Teichmüller metric may be defined on $T^{\#}(G)$.

The set of fixed points of an involutoric isometry of an open straight space is a nonempty open straight space [6]. We show $T'(H)$ is such a set.

LEMMA 2. *Let $h \in H$. Then $h \circ J = J \circ h_0$ for some $h_0 \in H$.*

PROOF. Let $h(z) = (az + b)/(cz + d) \in PSL(2, \mathbf{R})$. Then $h \circ J(z) = J \circ m(z)$ where $m(z) = (az - b)/(-cz + d)$ and $m \in PSL(2, \mathbf{R})$. Since $\rho \circ J = \bar{\rho}$, by the definition of H, $\rho \circ m(z) = g \circ \rho(z)$ for some $g \in G$. Hence $m = h_0 \in H$, and consequently $JHJ^{-1} = H$. Note also that $h_z \circ J(z) = \overline{(h_0)_z(z)}$. □

Let $\gamma \in M_1(H)$. Then $\bar{\gamma} \circ J \in M_1(H)$. As in the proof of Lemma 1, if w_γ has complex dilatation γ, the map $f(z) = (J \circ w_\gamma \circ J)(z)$ has complex dilatation $\bar{\gamma} \circ J$. If $w_\eta = w_\gamma$ on $\mathbf{R}$, $w_{\bar{\gamma} \circ J} = m \circ J \circ w_\gamma \circ J = m \circ J \circ w_\eta \circ J = w_{\bar{\eta} \circ J}$ on $\mathbf{R}$, where $m \in PSL(2, \mathbf{R})$ is the normalizer. Thus the map J induces the well-defined map $J^*: T(H) \to T(H)$ by $[w_\gamma] \to [w_{\bar{\gamma} \circ J}]$. J^* is clearly an involution since $\overline{(\bar{\gamma} \circ J)} \circ J = \gamma$, and it is also an isometry for

$$\left|\frac{\bar{\gamma}\circ J-\bar{\eta}\circ J}{1-(\eta\bar{\gamma})\circ J}\right|=\left|\frac{\bar{\gamma}-\bar{\eta}}{1-\eta\bar{\gamma}}\right|\circ J=\left|\frac{\gamma-\eta}{1-\bar{\eta}\gamma}\right|\circ J.$$

Since J is a homeomorphism, the essential supremums over U are equal. Hence J^* is (at most) distance decreasing. However, since $(J^*)^2$ is the identity, J must be an isometry.

PROPOSITION 2. *$T'(H)$ is an open straight space.*

PROOF. We show that $T'(H)$ is the set of fixed points of the isometric involution J^* of $T(H)$.

The equivalence class $[w_{\hat{\eta}}] \in T'(H)$ if and only if it contains an element w_η whose dilatation satisfies $\eta \circ J = \bar{\eta}$, or equivalently, $\bar{\eta} \circ J = \eta$. Each point of $T'(H)$ is thus clearly fixed by J^*.

Suppose conversely that $J^*([w_{\hat{\gamma}}]) = [w_{\hat{\gamma}}]$. Then $\bar{\hat{\gamma}} \circ J = \gamma' \sim \hat{\gamma}$. In particular, $[w_{\hat{\gamma}}]$ contains a Teichmüller mapping w_γ, and as such, w_γ has the strictly smallest dilatation. But $\|\gamma\| = \|\bar{\gamma} \cdot J\|$ which implies $[w_{\hat{\gamma}}] = [w_\gamma] \in T'(H)$. Thus $T'(H)$ is the fixed point set of J^*. □

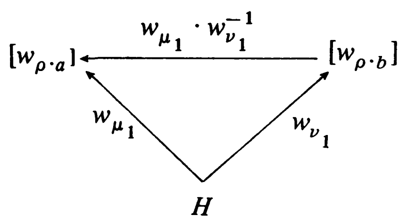
2. **The straightness of** $T^{\#}(G)$. We wish now to transfer the straight space structure of $T'(H)$ to $T^{\#}(G)$. The map $T^{\#}(G) \to T'(H)$ given by $\theta_\mu \to [w_{\rho\cdot\mu}]$ is one-to-one and onto. Suppose $w_{\rho\cdot\mu}$, $w_{\rho\cdot\nu}$ belong to $[w_{\rho\cdot a}]$, $[w_{\rho\cdot b}]$ respectively. Set $k(w_{\rho\cdot\mu} \circ w_{\rho\cdot\nu}^{-1})$ = dilatation of $w_{\rho\cdot\mu} \circ w_{\rho\cdot\nu}^{-1}$. Then

$$k\left(w_{\rho\cdot\mu}\circ w_{\rho\cdot\nu}^{-1}\right)=\|(\mu-\nu)/(1-\bar{\nu}\mu)\|$$

since ρ is onto. Thus

$$d^{\#}(\theta_a,\theta_b)\geqslant d\left([w_{\rho\cdot a}],[w_{\rho\cdot b}]\right),$$

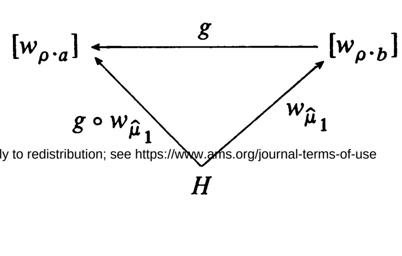
since the isomorphism $\theta_{\rho\cdot a}$, $\theta_{\rho\cdot b}$ may be induced by maps which are odd on $\mathbf{R}$ but do not commute with J on U. Consider



Suppose $k(w_{\mu_1} \circ w_{\nu_1}^{-1}) < k(w_{\rho\cdot\mu} \circ w_{\rho\cdot\nu}^{-1})$ for all $\rho\cdot\mu$, $\rho\cdot\nu \in M_1'(H)$ inducing $\theta_{\rho\cdot a}$, $\theta_{\rho\cdot b}$ respectively, where either μ_1 or $\nu_1 \notin M_1'(H)$.

Replace w_{ν_1} by $w_{\hat{\mu}_1} \sim w_{\nu_1}$ where $w_{\hat{\mu}_1}$ commutes with J. By Lemma 1, since $w_{\mu_1} \circ w_{\nu_1}^{-1}$ is odd on $\mathbf{R}$ we may replace it by an equivalent map g which satisfies $k(g) \leqslant k(w_{\mu_1} \circ w_{\nu_1}^{-1})$ and $g \circ J = J \circ g$ on U.

The diagram is then modified to



where $g \circ w_{\hat{\mu}_1} \sim w_{\mu_1}$ and $g \circ w_{\hat{\mu}_1}$ is symmetric. Then $k((g \circ w_{\hat{\mu}_1}) \circ w_{\hat{\mu}_1}^{-1}) = k(g) \leqslant k(w_{\mu_1} \circ w_{\nu_1}^{-1})$, a contradiction. The distance d will therefore be determined by Beltrami differentials in $M_1'(H)$, and the inequality is actually equality. The map $T^{\#}(G) \to T'(H)$ (and its inverse) is an isometry, and $T^{\#}(G)$ is an open straight space. We have proved

THEOREM. *Let G be a normalized finitely generated Fuchsian group. Then $T^{\#}(G)$ is an open straight space.*

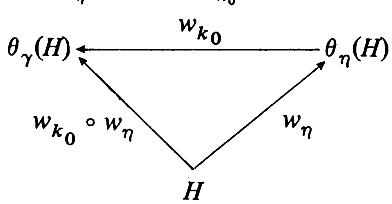
3. **A second characterization of** $T'(H)$. The equivalence class $[w_{\hat{\eta}}] \in T(H)$ is contained in $T'(H)$ if and only if $\theta_{\hat{\eta}}$ is induced by some w_η satisfying $w_\eta \circ J = J \circ w_\eta$. Let $\phi(z)dz^2 \in B_2'(H_\eta, U)$. The ray in $T(H)$ through $[w_{\hat{\eta}}]$ determined by ϕ lies entirely in $T'(H)$, for set $\eta_a(z) = a\bar{\phi}(z)/|\phi(z)|$, $0 \leqslant a < 1$. Then $\eta_a(Jz) = a\bar{\phi}(Jz)/|\phi(Jz)| = a\phi(z)/|\bar{\phi}(z)| = \bar{\eta}_a(z)$ and $\eta_a \in M_1'(H_\eta)$ for $0 \leqslant a < 1$. Hence w_{η_a} commutes with J, $w_{\eta_a} \circ w_\eta$ commutes with J, and $[w_{\eta_a} \circ w_\eta] \in T'(H)$.

On the other hand, suppose $[w_\eta]$, $[w_\gamma] \in T'(H)$. Then w_η, w_γ are odd one-to-one maps of $\mathbf{R}$ onto itself. Since $[w_\eta]$, $[w_\gamma]$ are also in $T(H)$, they lie on a unique straight line L in $T(H)$. In particular, they lie on the ray R of L determined by $\phi(z)dz^2$, $\phi \in B_2(H_\eta, U)$ where $k_0\bar{\phi}(z)/|\phi(z)|$, $0 \leqslant k_0 < 1$, is the dilatation of the (unique, extremal) Teichmüller map from $[w_\eta]$ to $[w_\gamma]$. We wish to show $\phi(z)dz^2 \in B_2'(H_\eta, U)$, for then this ray would lie in $T'(H)$. The same argument beginning with the inverse map $[w_\gamma] \to [w_\eta]$ will show that all of L lies in $T'(H)$.

The points of R are determined by maps w_k where

$$(w_k)_{\bar{z}}(z) = k\,\frac{\bar{\phi}(z)}{|\phi(z)|}\,(w_k)_z(z), \qquad 0 \leqslant k < 1, \phi \in B_2(H_\eta, U).$$

We assume w_η determines $\theta_\eta(H)$, and w_{k_0} determines $\theta_\gamma(H) = \theta_{k_0} \circ \theta_\eta(H)$.



Now $w_{k_0} \circ w_\eta$ and w_γ determine the same point in $T(H)$, hence they agree on $\mathbf{R}$. Thus

$$w_{k_0}(-x) = w_\gamma \circ w_\eta^{-1}(-x) = -\big(w_\gamma \circ w_\eta^{-1}\big)(x) = -w_{k_0}(x)$$

and w_{k_0} is odd on $\mathbf{R}$. Since w_{k_0} is uniquely extremal, by Lemma 1 it must be symmetric with respect to J.

For a symmetric w, $(w_{\bar{\zeta}} \circ J) = \bar{w}_z$, and $(w_\zeta \circ J) = \bar{w}_{\bar{z}}$. For $w = w_{k_0}$, if $w_{\bar{z}}(z) = k_0\bar{\phi}(z)w_z(z)/|\phi(z)|$, then

$$(w_{\bar{z}} \circ J)(z) = k_0\,\frac{(\bar{\phi} \circ J)(z)}{|(\phi \circ J)(z)|}\,(w_z \circ J)(z), \quad \bar{w}_z(z) = k_0\,\frac{(\bar{\phi} \circ J)(z)}{|(\phi \circ J)(z)|}\,\bar{w}_{\bar{z}}(z)$$

and consequently,

$$\frac{(\bar{\phi} \circ J)(z)}{|(\phi \circ J)(z)|} = \frac{\phi(z)}{|\bar{\phi}(z)|} .$$

Except for isolated zeros of ϕ, the function $(\bar{\phi} \circ J)(z)/\phi(z) = |(\phi \circ J)(z)/\bar{\phi}(z)|$ is conformal with constant zero imaginary part, and hence is constant. Thus $(\bar{\phi} \circ J)(z) = c\phi(z)$ for some $c \in \mathbf{R}$, $c > 0$. On the y axis, $J(y) = y$, and so $\bar{\phi}(y) = c\phi(y)$. Hence $c = 1$, $\phi \circ J = \bar{\phi}$, and $\phi(z)dz^2 \in B_2'(H_\eta, U)$. We have proved

Proposition 3. *The straight line connecting two points of $T'(H)$ lies in $T'(H)$, and is determined by symmetric quadratic differentials* ($\phi \circ J = \bar{\phi}$).

$T'(H)$ is a metric space when the Teichmüller metric is restricted to it. Let $\{\theta_n(H)\}$ be a bounded sequence of points of $T'(H)$. Since $\{\theta_n(H)\} \subset T(H)$, it must contain a subsequence converging to $\theta(H) \in T(H)$. But $\theta_n(H) \to \theta(H)$ if and only if the sequence of boundary values $\{w_n|_{\mathbf{R}}\} \to w|_{\mathbf{R}}$. By Lemma 1, $T'(H)$ is the set of equivalence classes of $T(H)$ with odd boundary values. Thus $[w] = \theta(H)$ must also be in $T'(H)$, and $T'(H)$ is finitely compact. The unique straight line in $T(H)$ connecting two points of $T'(H)$ lies in $T'(H)$; a second geodesic connecting two points of $T'(H)$ would also be one for $T(H)$, contradicting the uniqueness there. We have again proved

Proposition 2′. *$T'(H)$ is an open straight space.*

Proposition 3 also allows us to complete the proof of Teichmüller's theorem for $T^{\#}(G)$.

Proposition 4. *The extremal element of θ_μ is a Teichmüller mapping.*

Proof. Let w_μ be the unique extremal element of $\theta_\mu \in T^{\#}(G)$. Then $\rho \cdot \mu \in M_1'(H)$ is by Proposition 1, the dilatation of the unique extremal element of $[w_{\rho\cdot\mu}]$, which is symmetric with respect to J. By Teichmüller's theorem for $T(H)$, $(\rho \cdot \mu)(z) = k\bar{\phi}(z)/|\phi(z)|$ for some k, $0 \leqslant k < 1$, $\phi \in B_2(H, U)$. But ϕ determines a ray between two points of $T'(H)$ (from $\theta_{\mathrm{id}}(H) = (H)$ to $\theta_{\rho\cdot\mu}(H)$) and therefore belongs to $B_2'(H, U)$ by Proposition 3. But $\phi = \psi \times \rho$ where $\psi \in B_2(G,\Omega)$, and

$$(\rho \cdot \mu)(z) = k\,\frac{\bar{\phi}(z)}{|\phi(z)|} = k\,\frac{(\overline{\psi \times \rho})(z)}{|(\psi \times \rho)(z)|} = \rho \cdot \left(k\,\frac{\bar{\psi}}{|\psi|}\right)(z)$$

which implies that $\mu = k\bar{\psi}/|\psi|$. Since $\psi \in B_2(G, \Omega)$, w_μ is a Teichmüller mapping, and we have completed the proof of Teichmüller's theorem for finitely generated normalized Fuchsian groups of the second kind.

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